Online Appendix (for online publication only)

In this note, we prove all claims we made regarding FB in the main context and demonstrate that dynamic consistency implies updating monotonicity.

Results on FB

For any given preference $\succsim$ in $\mathscr{R}$ and event E that is $\succsim$-non-null, denote by $\succsim_{E,fb}$ and $\succsim_{E,\Gamma}$ the ex-post preferences of $\succsim$ updated with FB and a given updating rule Γ, respectively, when E occurs. Recall that for all $P \in \mathscr{P}$ and P-non-null event E, the FB posterior set is given by $Q^{fb}(P,E) = cl\Big(\{\overline{p|E} : p \in P, p(E) > 0\}\Big)$ which is an element in $\mathscr{P}$. For event E, we say that E is strictly $\succsim$-non-null if for all f, g satisfying $g(s) \succsim f(s)$ for all $s \in S$ and $g(s') \succ f(s')$ for all $s' \in E$, we have $f \succ g$. When $\succsim$ is represented by (u, P), it can be shown that E is strictly $\succsim$-non-null if and only if $\min_{p\in P} p(E) > 0$. Denote by $\mathcal{S}^*(\succsim)$ the set of all strictly non-null events of $\succsim$. When $\min_{p\in P} p(E) > 0$, the set $\{\overline{p|E} : p \in P\}$ is closed and thus $Q^{fb}(P,E) = \{\overline{p|E} : p \in P\}$. Throughout the proof, let Π denote the partition $\{\{s\} : s \in E\} \cup \{S\backslash E\}$ whenever E is clearly specified. For a given preference that is represented by (u, P), we simply say that it is represented by P.

Proposition A1. FB satisfies Alignment Consistency*.

Proof. Consider preferences $\{\succsim^k\}_{k=1}^3$ and event E that satisfy the conditions in the statement of the Alignment Consistency* axiom. Let the three preferences be represented by P^1, P^2 and P^3, respectively. Let $Q = Q^{fb}(P^1,E) = Q^{fb}(P^2,E)$. Since $\succsim^3$ is E-aligned with $(\succsim^1, \succsim^2)$, by Lemma 3, P^3 is E-aligned with (P^1, P^2). Thus, we have $P^3_\Pi \subseteq co(P^1_\Pi \cup P^2_\Pi)$. It follows that $Q^{fb}(P^3,E) \subseteq Q^{fb}(co(P^1_\Pi \cup P^2_\Pi),E) = Q$. To see that $Q \subseteq Q^{fb}(P^3,E)$, consider any $q \in Q$. By the definition of FB, there exists a sequence $(p^{1,k})_{k=1}^{+\infty}$ in $\{p \in P^1 : p(E) > 0\}$ and a sequence $(p^{2,k})_{k=1}^{+\infty}$ in $\{p \in P^2 : p(E) > 0\}$ such that both $(\overline{p^{1,k}|E})_{k=1}^{+\infty}$ and $(\overline{p^{2,k}|E})_{k=1}^{+\infty}$ converge to q. By the E-alignment relation between P^3 and (P^1,P^2), for each $k \in \mathbb{N}_+$, we can find $\alpha_k \in [0,1]$ such that $\alpha_k p^{1,k}_\Pi + (1-\alpha_k)p^{2,k}_\Pi \in P^3_\Pi$. It follows that $(\overline{(\alpha_k p^{1,k} + (1-\alpha_k)p^{2,k})|E})_{k=1}^{+\infty}$ converges to q. Thus, $Q \subseteq Q^{fb}(P^3,E)$, and we conclude that $Q = Q^{fb}(P^3,E)$, i.e., $\succsim^1_{E,fb} = \succsim^2_{E,fb} = \succsim^3_{E,fb}$. □

Proposition A2. FB satisfies Sensitivity Independence*.

Proof. Consider preferences $\succsim^1$ and $\succsim^2$, event E, and $\lambda \in [1,+\infty)$ that satisfy the conditions in the statement of the Sensitivity Independence* axiom. Let $\succsim^1$ and $\succsim^2$ be represented by P^1 and P^2 respectively. By Lemma 5, we have $\lambda P^1|E = P^2|E$, and it follows that $Q^{fb}(P^1,E) = Q^{fb}(P^2,E)$. Therefore, we have $\succsim^1_{E,fb} = \succsim^2_{E,fb}$. □

Propositions A1 and A2 imply that FB satisfies Alignment Consistency, Sensitivity Congruence, and Sensitivity Independence.

Proposition A3. FB satisfies Increased Sensitivity after Updating.

Proof. Consider $\succsim \in \mathscr{R}$ and $E \in \mathcal{S}(\succsim)$ such that $\succsim_{E,fb}$ is unambiguous. Let $\succsim$ and $\succsim_{E,fb}$ be represented by (u, P) and $(u, \{q\})$, respectively. It follows that for all $p \in P$ with $p(E) > 0$, $\overline{p|E} = q$. Therefore, for all $s \in E$ and $f, g \in \mathcal{F}$ with $f \overset{S\backslash s}{=} g$ and $g(s) \succ f(s)$, we have $u^{\downarrow}(g; P) - u^{\downarrow}(f; P) \leq \max_{p\in P} p(s)(u(g(s)) - u(f(s))) \leq q(s)(u(g(s)) - u(f(s))) = u(g; q) - u(f; q)$. Therefore, the axiom holds for FB. □

The next proposition states that FB generically satisfies Continuity.

Proposition A4. For all $\succsim \in \mathscr{R}$, sequence $(\succsim^k)_{k=1}^{+\infty}$ in $\mathscr{R}$, and $E \in \mathcal{S}^*(\succsim) \cap (\cap_{k=1}^{+\infty} \mathcal{S}(\succsim^k))$, if $(\succsim^k)_{k=1}^{+\infty}$ converges to $\succsim$ on E, then $(\succsim^k_{E,fb})_{k=1}^{+\infty}$ converges to $\succsim_{E,fb}$ on E.

Proof. Let $\succsim$ be represented by P, and for every $k \in \mathbb{N}$, let $\succsim^k$ be represented by P^k. Since $(\succsim^k)_{k=1}^{+\infty}$ converges to $\succsim$ on E, by Lemma 7, $(P^k_\Pi)_{k=1}^{+\infty}$ converges to P_Π. Since $E \in \mathcal{S}^*(\succsim)$, there exists $\lambda \in (0, 1)$ such that $\min_{p\in P} p(E) \geq \lambda$. Therefore, it is WLOG to assume that there exists $\lambda^* \in (0, 1)$ such that $\min_{p\in P} p(E) \geq \lambda^*$, and for all k, $\min_{p\in P^k} p(E) \geq \lambda^*$. Thus, to show that $(\succsim^k_{E,fb})_{k=1}^{+\infty}$ converges to $\succsim_{E,fb}$ on E, it suffices to show that $(\{\overline{p|E} : p \in P^k\})_{k=1}^{+\infty}$ converges to $\{\overline{p|E} : p \in P\}$, which follows from the fact that $(P^k_\Pi)_{k=1}^{+\infty}$ converges to P_Π. □

Our final proposition states that FB can be characterized by Alignment Consistency* and Sensitivity Independence* in the (generic) situation where the realized event is strictly non-null with respect to the ex-ante preference.

Proposition A5. If an updating rule Γ satisfies Alignment Consistency* and Sensitivity Independence*, then for all $\succsim \in \mathscr{R}$ and $E \in \mathcal{S}^*(\succsim)$, $\succsim_{E,fb} = \succsim_{E,\Gamma}$.

Proof. Consider $\succsim \in \mathscr{R}$ and $E \in \mathcal{S}^*(\succsim)$, and let $\succsim$ be represented by P. Consider the non-trivial case in which $p(E) < 1$ for some $p \in P$. Since $E \in \mathcal{S}^*(\succsim)$, we have $Q^{fb}(P, E) = \{\overline{p|E} : p \in P\}$. Let $\succsim^1 \in \mathscr{R}$ be the preference that is represented by $Q^{fb}(P, E)$. Let $\succsim^2 \in \mathscr{R}$ be the preference that is represented by $\alpha Q^{fb}(P, E) + (1-\alpha)\{q^*\}$, where q^* satisfies $q^*(s^*) = 1$ for some $s^* \in S\backslash E$, and $\alpha \in (0, 1)$ satisfies $\alpha < \min_{p\in P} p(E)$. It follows that $\succsim^2 \overset{(\frac{1}{\alpha}, E)}{\rightsquigarrow} \succsim^1$, and by Sensitivity Independence*, we have $\succsim^1_{E,\Gamma} = \succsim^2_{E,\Gamma}$. Since $\succsim^1_{E,\Gamma} = \succsim^1$, we have $\succsim^2_{E,\Gamma} = \succsim^1$. By Alignment Consistency*, it suffices to show that $\succsim$ is E-aligned with $(\succsim^1, \succsim^2)$. By Lemma 3, we need to show that P is E-aligned with $(\alpha Q^{fb}(P, E) + (1-\alpha)\{q^*\}, Q^{fb}(P, E))$. Since $\alpha < \min_{p\in P} p(E)$, for every $p^* \in P$, we have $p^*_\Pi \in co(\{\overline{p^*|E}, \alpha\overline{p^*|E} + (1-\alpha)q^*\})_\Pi$. Therefore, $P_\Pi \subseteq co\Big(Q^{fb}(P, E) \cup \Big(\alpha Q^{fb}(P, E) + (1-\alpha)\{q^*\}\Big)\Big)_\Pi$. Next, consider arbitrary $q^1 \in Q^{fb}(P, E)$ and $r \in \alpha Q^{fb}(P, E) + (1-\alpha)\{q^*\}$. Let $r = \alpha q^2 + (1-\alpha)q^*$ for some $q^2 \in Q^{fb}(P, E)$. For q^1 and q^2, we can find $p^1, p^2 \in P$ such that $q^1 = \beta^1 p^1|E$ and $q^2 = \beta^2 p^2|E$ for some $\beta^1, \beta^2 \in [1, +\infty)$. Since $\alpha < \min_{p\in P} p(E)$, we have $\beta^2 < \frac{1}{\alpha}$. Let $t = \frac{1-\alpha\beta^2}{\beta^1-\alpha\beta^2} \in [0, 1]$, and it follows that $tq^1 + (1-t)r = t\beta^1 p^1|E + (1-t)\alpha\beta^2 p^2|E + (1-t)(1-\alpha)q^*$ satisfies $t\beta^1 + (1-t)\alpha\beta^2 = 1$. Therefore, $(tq^1 + (1-t)r)_\Pi = (t\beta^1 p^1 + (1-t)\alpha\beta^2 p^2)_\Pi \in P_\Pi$. □

Updating Monotonicity and Dynamic Consistency

In this section, we demonstrate that under mild conditions, dynamic consistency implies updating monotonicity. Consider a preference $\succsim$ over $\mathcal{F}$ and an event E. Let $\succsim_E$ be another preference over $\mathcal{F}$ such that $S \backslash E$ is $\succsim_E$-null. We interpret $\succsim_E$ as the ex-post preference updated from $\succsim$ when E occurs. We do not require $\succsim$ or $\succsim_E$ to be maxmin preferences. Instead, we only require that $\succsim$ and $\succsim_E$ satisfy Axioms M1, M2 and M4.

The tuple $(\succsim, \succsim_E)$ is said to satisfy updating monotonicity on E if for all $s \in E$, $x, y \in X$, and $f, g \in \mathcal{F}$, if $f \overset{S \backslash s}{=} g$, $g(s) \succ f(s)$, $f \sim x$, $f \sim_E x$ and $g \sim y$, then $g \succsim_E y$. We say that $(\succsim, \succsim_E)$ satisfies dynamic consistency on E if for all $f, g, h \in \mathcal{F}$, $fEh \succsim gEh$ implies $fEh \succsim_E gEh$.

Proposition A6. Assume that $\succsim$ and $\succsim_E$ satisfy Axioms M1, M2 and M4. If $(\succsim, \succsim_E)$ satisfies dynamic consistency on E, then $(\succsim, \succsim_E)$ satisfies updating monotonicity on E.

Proof. Consider $s \in E$, $x, y \in X$ and $f, g \in \mathcal{F}$ such that $f \overset{S \backslash s}{=} g$, $g(s) \succ f(s)$, $f \sim x$, $f \sim_E x$ and $g \sim y$. We want to show that $g \succsim_E y$. By Axioms M1 and M4, we have $g \succsim f$ and $y \succsim x$. Since $f \sim x$ and $g \sim y$, by Axioms M1 and M2, we have $\frac{1}{2}f + \frac{1}{2}y \sim \frac{1}{2}g + \frac{1}{2}x$. Consider act h such that $h = (\frac{1}{2}f + \frac{1}{2}y)E(\frac{1}{2}g + \frac{1}{2}x)$. By $f \overset{S \backslash s}{=} g$, $y \succsim x$ and Axiom M2, we have for all $s \in S$, $(\frac{1}{2}f + \frac{1}{2}y)(s) \succsim h(s)$. Therefore, we have $\frac{1}{2}f + \frac{1}{2}y \succsim h$, and thus $\frac{1}{2}g + \frac{1}{2}x \succsim h$. By dynamic consistency, we have $\frac{1}{2}g + \frac{1}{2}x \succsim_E h$. Since $S \backslash E$ is $\succsim_E$-null, $\frac{1}{2}g + \frac{1}{2}x \succsim_E h$ implies $\frac{1}{2}g + \frac{1}{2}x \succsim_E \frac{1}{2}f + \frac{1}{2}y$. Since $f \sim_E x$, by Axioms M1 and M2, we have $g \succsim_E y$. □