A Theory of Revealed Indirect Preference*

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Abstract

We call a preference over menus an *indirect preference* if there exists a preference over the objects that make up the menus, and a menu is ranked over another whenever it contains an object that is preferred to every object in the other menu. Suppose an observer has information on an agent's ranking over some menus; we characterize those rankings that guarantee the existence of a preference over objects that induces the observed menu rankings. Our result has the following applications. (1) It gives a characterization of rankings over prices that could be extended to a bona fide indirect utility function. (2) It leads to a generalization of Afriat's Theorem that allows for imperfectly observed choices. (3) It leads to a test of the multiple preferences model. (4) It helps us characterize a model of choice generated by minimax regret.

KEYWORDS: Indirect utility, preference over prices, imperfect observations, multiple rationales, minimax regret

1 Introduction

This paper explores the structure of indirect preference. Given a set of alternatives X, we refer to nonempty subsets of X as menus. A preference over menus constitutes an *indirect preference* so long as there is a preference over the alternatives in X such that menu A is preferred to another menu B whenever A contains an object that is preferred to every object in B. The study of indirect preferences has a long

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history in economic theory. Indeed, a basic question in consumer theory concerns the recovery of the direct utility function (defined on bundles of n goods) from the indirect utility over the prices of those goods. In this case, plainly, a vector of prices (with income held fixed at some value) corresponds to a linear budget set, which is just a specific type of menu from the consumption space $X = \mathbb{R}^n_+$. It is well-known that the crucial property that makes it possible for a function defined over prices to be a bona fide indirect utility function is for it to be quasiconvex in prices (see Krishna and Sonnenschein (1990) and Jackson (1986)).

This question could be posed in a more general form that is not specific to the consumer theory context. Kreps (1979) considers *all* possible menus drawn from a set of alternatives X and shows that a preference over these menus constitutes an indirect preference if and only if it satisfies the following property: an agent who prefers menu A to B will be indifferent between A and $A \cup B$.[1] Tyson (2018) extends this result by characterizing indirect preference defined over a given *subcollection* of menus.

In this paper, we consider an observer who has access to a finite collection of observations, $\mathcal{M} := \left\{(A^t, B^t)\right\}_{t\in T}$. At each observation t, the observer knows that the agent weakly prefers menu A^t to B^t; and for a (possibly empty) subcollection S of those observations, the observer knows that A^t is strictly preferred to B^t for each $t \in S$. We say that the data set $\mathcal{M}$ can be *rationalized* if the observed preference between each pair of menus is part of an indirect preference over menus (induced by a direct preference over the elements of X).

Our first result establishes that a data set $\mathcal{M}$ can be rationalized if and only if it satisfies the *never-covered property*; this is an intuitive property that could be defined via an iterative procedure. As a simple example of what it entails, suppose the observer knows that the agent strictly prefers menu A to B and weakly prefers A' to B'. Since there is an alternative in A that strictly dominates everything in B, a necessary condition for rationalization is that B does not contain A, i.e., $A \setminus B \neq \emptyset$. But this is not all. We also need to check if $A' \subseteq B$; if this holds, then clearly there is some element in A that dominates every element in $B \cup B'$ and so $A \setminus (B \cup B')$ must also be nonempty. In other words, some element must remain in A after all of A's revealed dominated elements have been iteratively excluded; this property turns out to be both necessary and sufficient for rationalization.

[1] Kreps (1979) uses this as a benchmark for the axiomatization of preference for flexibility. Other models of menu preferences include (among others) Dekel et al. (2001), Gul and Pesendorfer (2001), Dekel et al. (2009) and Dekel and Lipman (2012).

Note that the setting of our result is different from that of Kreps (1979), Tyson (2018), or the characterization results on indirect utility over prices. In those papers, it is assumed that the observer knows the *complete* ranking over the collection of menus under consideration. In contrast, our result allows the observer to have only an incomplete ranking over menus; for example, the observer need not know the agent's preference between menus A^t and $A^{t'}$. The never-covered property reduces to the properties found in the earlier papers if the set of observations leads to a complete ranking over the collection of menus being considered.

Another paper that is closely related to ours is that of Fishburn (1976). That paper studies a different rationalizability problem (closely connected to the issue of coarse rationalizability discussed below), but its basic result could be interpreted as a result on preference over menus; interpreted in that way, it provides a characterization of indirect preference (through a property called the *partial congruence axiom*) in the special case where A^t is *strictly* preferred to B^t for each $t \in T$. Our never-covered property is a generalization of the partial congruence axiom that allows for the possibility that the observer only knows that one menu is *weakly* preferred to another. This generalization is nontrivial and it is also crucial in certain applications, including the application to coarse rationalizability discussed below.

In some applications, it may not suffice to have a preference on X that rationalizes a data set $\mathcal{M}$; it may also be desirable to have the preference be the extension of some given preorder. For example, in the case of consumer theory, it would be natural to require any rationalizing preference to be increasing in the product order on the consumption space $X = \mathbb{R}^n_+$. In cases where the space of alternatives X, and the menus defined on it, contain infinitely many elements, it is also natural to assume that there is a topology on X and to require preferences over X to be continuous, which guarantees the existence of optimal elements when menus are compact. We show that our basic result can be extended to incorporate these features. Last but not least, we show that the never-covered property can be verified via an efficient algorithm, which facilitates the empirical application of our results.

The paper also discusses four applications of our theory.

(1) First, we revisit the question of characterizing indirect utility over prices. Instead of assuming that the entire indirect utility function is known, we assume that the observer only knows the consumer's preference for a finite set of price pairs, i.e., p^t is preferred to q^t (for $t = 1, 2, \dots, T$), with income normalized at 1. We show that there is an increasing, continuous and concave utility function that rationalizes the observed price preferences provided the latter satisfies a generalization of the quasiconvex property (on indirect utility functions).

(2) Consider a data set with T observations; at each observation t, it is observed that an agent chooses x^t from the menu C^t. Various versions of Afriat's Theorem (1967)[2] answer the following question: what necessary and sufficient conditions on $\mathcal{O} = \{(x^t, C^t)\}_{t\in T}$ guarantee the existence of a utility function U defined on X such that $x^t \in \arg\max_{x\in C^t} U(x)$ for all observations t? When $\mathcal{O}$ is not rationalizable in this sense, it would be natural to relax the rationalizability requirement by allowing the optimal choice to live in a ball containing x^t, and the required size of this ball could then be used as a measure of how close $\mathcal{O}$ is to being rationalizable.

For this reason and for others, it is useful to study *coarse data sets*, where at observation t, the agent's choice is not precisely specified but is known only to come from a set $A^t \subseteq C^t$; given this, rationalization requires that there be a utility function U such that $A^t \cap \arg\max_{x\in C^t} U(x)$ is nonempty at each t. This is equivalent to the condition that (considered as menus) A^t is weakly preferred to C^t for all t. It follows that the never-covered property could be used to characterize the rationalizability of coarse data sets and our algorithm provides a way of checking for this property in empirical applications.[3]

In the Online Appendix, we study data from an experiment where subjects choose consumption bundles from budget sets. We apply our algorithm to calculate, for each subject, a *perturbation index* that measures the extent to which her consumption choices have to be perturbed to guarantee rationalizability. These calculations illustrate the ease with which our algorithm can be implemented.

The last two applications of our theory provide characterizations of choice models outside the classical paradigm.

(3) In the multiple preferences model (see Aizerman and Malishevski (1981), Moulin (1985), and Salant and Rubinstein (2008)), an agent's choice from a menu C could be the optimal choice for any one of a *set* of preferences. The question we pose is the following: suppose that for a finite collection of menus C^t we observe the agent's choices from each menu, which we denote by A^t; when can we find a collection

[2] See, for example, Varian (1982), Forges and Minelli (2009), Reny (2015), and Nishimura et al. (2017). For a textbook treatment of Afriat's Theorem, see Kreps (2013) and Chambers and Echenique (2016).

[3] Fishburn (1976) considers a related problem which requires $\arg\max_{x\in C^t} U(x) \subseteq A^t$ for all observations t. This is equivalent to requiring the menu A^t to be strictly preferred to $C^t \setminus A^t$ for all t. We think that our formulation of rationalizability is more appropriate in empirical applications; unlike Fishburn's formulation, we allow for the possibility that an alternative in $C^t \setminus A^t$ is also optimal for the agent and it is the natural generalization of the rationalizability notion in Afriat's Theorem, which does allow for the optimality of alternatives in $C^t \setminus \{x^t\}$.

of preferences $\{\succsim_i\}_{i\in I}$ such that $Z^t = A^t$, where Z^t consists of all elements of C^t that are optimal according to some preference $\succsim'$ in $\{\succsim_i\}_{i\in I}$? It turns out that this problem can be reformulated as a problem of rationalizing menu preferences and thus could be solved with the never-covered property. We provide a characterization of rationalizability with multiple preferences and also an algorithm for checking this characterization. Our result generalizes the finding in Aizerman and Malishevski (1981), which addresses this issue in the case where the choice set from *every* possible menu is observed.

(4) The second model we consider is the minimax regret model (Wald, 1950; Savage, 1951). In this model, an agent has multiple utility functions over alternatives, drawn from a set U. For a given menu A, the regret of alternative x under one of the agent's utility functions $u \in U$ is given by $\max_{y\in A} u(y) - u(x)$. The agent evaluates alternative x according to its maximal regret $\max_{u\in U} (\max_{y\in A} u(y) - u(x))$, and chooses alternatives from the choice set that minimize the maximal regret. Using the never-covered property once again, we could find necessary and sufficient conditions on a data set under which the set of choices at each observed menu coincides exactly with the set of model consistent choices, for an appropriately chosen U.

Organization of paper. In Section 2, we set out the basic definitions used throughout the paper. In Section 3, we introduce the main theorem, discuss several of its special cases, and also formulate the algorithm to test the never-covered property. The four applications of our theory are presented (respectively) in Sections 4, 5, 6, and 7. All omitted proofs are in the Appendix. There is also an Online Appendix containing secondary results on the never-covered property as well as an illustrative implementation of our algorithm for checking that property.

2 Preliminaries

We work with a fixed nonempty set X, which can be viewed as the universal set of alternatives. Let $\mathcal{X}$ denote the collection of nonempty subsets of X. We refer to elements of $\mathcal{X}$ as menus. Generic elements of X are denoted by x, y, z, etc, while generic elements of $\mathcal{X}$ are denoted by A, B, C, etc.

A preorder $\trianglerighteq$ on X is a binary relation on X that is reflexive and transitive.[4] We use $\triangleright$ to denote the asymmetric part of $\trianglerighteq$. For a given preorder $\trianglerighteq$ on X and a menu $A \in \mathcal{X}$, we define $A^{\downarrow}$ to be the decreasing closure of A with respect to the preorder $\trianglerighteq$, i.e.,

$$A^{\downarrow} := \{x \in X : y \trianglerighteq x \text{ for some } y \in A\},$$

and define $A^{\downarrow\downarrow}$ to be the strictly decreasing closure of A with respect to the preorder $\trianglerighteq$, i.e.,

$$A^{\downarrow\downarrow} := \{x \in X : y \triangleright x \text{ for some } y \in A\}.$$

The set of $\trianglerighteq$-undominated alternatives in A is denoted by $\max(A; \trianglerighteq) := A \setminus A^{\downarrow\downarrow}$.

A *preference* $\succsim$ on X is a complete preorder on X. We use $\succ$ to denote the asymmetric part of $\succsim$. When $\succ$ denotes the asymmetric part of an anti-symmetric preference, we refer to it as a *strict preference*. We say that x is a $\succsim$-maximal element in A if $x \in A$ and $x \succsim y$ for all $y \in A$, and write $\max(A; \succsim)$ to denote the set of $\succsim$-maximal alternatives in A. When convenient, we write $x \succsim A$ if $x \succsim y$ for each $y \in A$ and $x \succ A$ if $x \succ y$ for each $y \in A$.

Oftentimes, it is useful to study preferences restricted to a particular class. We say that the preference $\succsim$ extends the preorder $\trianglerighteq$ if

$$x \succsim y \text{ whenever } x \trianglerighteq y, \text{ and } x \succ y \text{ whenever } x \triangleright y.$$

We can think of $\trianglerighteq$ as an exogenously given dominance relation on X, and view the statement $x \trianglerighteq y$ as saying that x is an objectively better alternative than y, which the agent's preference $\succsim$ should respect.

The general choice environment we define here follows that in Nishimura et al. (2017). As a basic example of this environment, we note that in consumer theory, the consumption space with n commodities is typically $X = \mathbb{R}^n_+$. Furthermore, if a consumer always strictly prefers to have more of any good, the the consumer's preference would extend the coordinate-wise or product order $\geq$ on $\mathbb{R}^n_+$.[5] Bear in mind that our setup also allows for a preference without any preorder restrictions: in this case, the preference extends the trivial preorder $\trianglerighteq$ where $x \trianglerighteq y$ only if $x = y$.

[4] Terminology: a binary relation R on X is a nonempty subset of $X \times X$, but as usual, we write xRy instead of $(x, y) \in R$. We say that R is reflexive if xRx for each $x \in X$, transitive if xRy and yRz imply xRz for each $x, y, z \in X$, complete if either xRy or yRx holds for any $x, y \in X$, and anti-symmetric if for any $x, y \in X$ such that $x \neq y$, xRy and yRx do not hold simultaneously. The asymmetric part of R is defined as the binary relation P on X such that xPy if and only if xRy but not yRx.

[5] Formally, $x \geq y$ if and only if $x_i \geq y_i$ for each $i \in \{1, 2, \ldots, n\}$, and $x > y$ if and only if $x \geq y$ and $x_i > y_i$ for some $i \in \{1, 2, \ldots, n\}$. We write $x \gg y$ if and only if $x_i > y_i$ for each $i \in \{1, 2, \ldots, n\}$.

3 Rationalizability of menu preferences

In this section, we study the conditions under which a finite list of observed menu preference pairs collected from an agent is consistent with some (unobserved) preference on the underlying alternatives. The data collected by the observer is formally represented as $\mathcal{M} := \left\{(A^t, B^t)\right\}_{t\in T}$, where T is a nonempty finite index set and A^t and B^t are menus. For each t, the observer either knows that the agent weakly prefers A^t to B^t or that the agent strictly prefers A^t to B^t.[6] Let W be the collection of observations where A^t is weakly preferred to B^t, and let S be the collection of observations where A^t is strictly preferred to B^t. By definition, $\{W, S\}$ is a partition of T. For any nonempty $T' \subseteq T$, we let

$$A(T') := \textstyle\bigcup_{t\in T'} A^t \text{ and } B(T') := \bigcup_{t\in T'} B^t.$$

The following definition specifies precisely what it means for $\mathcal{M}$ to be rationalized.

Definition 1. *A set of menu preference pairs $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ is* rationalized *by a preference $\succsim$ on X if for any $t \in T$, there exists $x^t \in A^t$ such that*

(1) $x^t \succsim B^t$ *and*

(2) $x^t \succ B^t$ *if* $t \in S$.

In this case, we say that $\mathcal{M}$ is rationalizable. *A preference $\succsim$* $\trianglerighteq$-rationalizes *$\mathcal{M}$ if $\succsim$ rationalizes $\mathcal{M}$ and extends $\trianglerighteq$; in this case, $\mathcal{M}$ is* $\trianglerighteq$-rationalizable.[7]

Our objective in this section is to characterize those sets of menu preference pairs $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ which can be $\trianglerighteq$-rationalized. Readers familiar with the revealed preference theory will notice that the issue is fairly straightforward when A^t is a singleton for each t. In that case, the problem is (in its essentials) within the scope of the well-known theorems of Afriat (1967) and Richter (1966) and their extensions; in the manner of those theorems, some version of a no-cycling condition on the

[6] This formulation includes the case in which the observer knows that the agent is indifferent between two menus (say) A and B, because this case could be considered as two observations, with the agent weakly preferring A to B in one observation and B to A in the other observation.

[7] Definition 1 is one of several possible formulations of the rationalizability of a set of menu preference pairs. For example, one could require that for any $t \in T$, (1) for any $y \in B^t$, there exists $x \in A^t$ such that $x \succsim y$ and (2) there exists $x^t \in A^t$ such that $x^t \succ B^t$ if $t \in S$. To see the (subtle) differences between these two formulations, suppose that the preorder is $\geq$. The first data set has only one observation (B, B) where $B = (0, 1)$ and the relation is weak. The second data set has only one observation (A, B) where A is the set of rational numbers in $(0, 1)$, $B = (0, 1)$, and the relation is weak. Definition 1 would classify both data sets as not $\geq$-rationalizable, while the alternative formulation would classify both as $\geq$-rationalizable. That being said, in all the applications that we study (and in most economic environments), either X is finite or X is infinite but the menus are compact. In these cases, these two formulations are equivalent.

revealed preference relations defined on $\{x^t\}_{t\in T}$ (which will be stated formally later in Definition 3) is both necessary and sufficient for the existence of a preference that extends $\trianglerighteq$ and satisfies (1) and (2) in Definition 1.[8] Of course, A^t is typically not a singleton. Thus, we could understand the issue before us in the following way: we have to formulate a property on $\mathcal{M}$ that guarantees *the existence of a selection* x^t from A^t, such that the resulting (notional) set of observations $\left\{(\{x^t\}, B^t)\right\}_{t\in T}$ satisfies the required no-cycling condition. The next subsection provides the property guaranteeing that such a selection exists.

3.1 The never-covered property

Suppose that the data set $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ is $\trianglerighteq$-rationalized by some preference relation $\succsim$. Consider an arbitrary observation t. By the definition of $\trianglerighteq$-rationalizability, A^t contains an alternative x with $x \succsim B^t$. Since $\succsim$ extends $\trianglerighteq$, $x \notin B^{t\downarrow\downarrow}$. Moreover, if $t \in S$, then A^t contains an alternative x with $x \succ B^t$ and so $x \notin B^{t\downarrow}$. Thus A^t cannot be covered by (in other words, contained in) $B^{t\downarrow\downarrow}$ if $t \in W$, and cannot be covered by $B^{t\downarrow}$ if $t \in S$.

This argument could be generalized to more than one observation. For any nonempty subset $T' \subseteq T$, notice that (1) if x satisfies $x \succsim B(T')$ then $x \notin B(T')^{\downarrow\downarrow}$; and (2) if x satisfies $x \succ B(T' \cap S)$ then $x \notin B(T' \cap S)^{\downarrow}$. Thus, if $A(T')$ contains an alternative $\hat{x}$ satisfying both conditions, then $A(T')$ cannot be covered by $B(T')^{\downarrow\downarrow} \bigcup B(T' \cap S)^{\downarrow}$. And we can indeed find such an alternative $\hat{x}$ in $A(T')$: for each $t \in T'$, pick $x^t \in A^t$ such that $x^t \succsim B^t$ if $t \in W$ and $x^t \succ B^t$ if $t \in S$; then let $\hat{x} \in \max(\{x^t\}_{t\in T'}; \succsim) \in A(T')$. Let $\hat{t}$ be an observation at which $\hat{x} \in A^{\hat{t}}$.

We are now ready to introduce the procedure that we call *the iterated exclusion of dominated observations*. Given a nonempty subset of observations T', let $\Phi^0(T') := \emptyset$, and let $\Phi^1(T')$ be the collection of observations t such that A^t is covered by $B(T')^{\downarrow\downarrow} \bigcup B(T' \cap S)^{\downarrow}$, i.e.,

$$\Phi^1(T') := \left\{t \in T' : A^t \subseteq B(T')^{\downarrow\downarrow} \bigcup B(T' \cap S)^{\downarrow}\right\}.$$

Since $\hat{t} \notin \Phi^1(T')$, we have $\Phi^1(T') \neq T'$. Let

$$\Phi^2(T') := \left\{t \in T' : A^t \subseteq B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi^1(T'))^{\downarrow}\right\}.$$

[8] To be precise, Nishimura et al. (2017) already provides a condition on $\left\{(\{x^t\}, B^t)\right\}_{t\in T}$ that is necessary and sufficient for the existence of a preference $\succsim$ that extends $\trianglerighteq$ and satisfies $x^t \succsim B^t$ for each $t \in T$. In our case, we potentially have observations where we require $x^t \succ B^t$; thus a modification of the condition in Nishimura et al. (2017) is required to accommodate these cases, but this extension of their result is fairly straightforward.

Obviously, $\Phi^1(T') \subseteq \Phi^2(T')$. Since $\hat{x} \succsim B(T')$ and $\hat{x} \succ B(T' \cap S)$, we obtain $\hat{x} \succ A^t$ for each $t \in \Phi^1(T')$; since A^t is preferred to B^t, we know that $\hat{x} \succ B^{t\downarrow}$ for each $t \in \Phi^1(T')$. Thus, $\hat{x} \succ B((T' \cap S) \cup \Phi^1(T'))^{\downarrow}$. We conclude that $A(T')$ (which contains $\hat{x}$) cannot be covered by $B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi^1(T'))^{\downarrow}$ and so $\hat{t} \notin \Phi^2(T')$. We may repeat this argument for $m = 2, 3, \ldots,$ where

$$\Phi^{m+1}(T') := \left\{t \in T' : A^t \subseteq B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi^m(T'))^{\downarrow}\right\}.$$

Since $\Phi^m(T')$ is an increasing sequence in m in the set inclusion sense and T' is finite, the procedure stops at m^* when $\Phi^{m^*}(T') = \Phi^{m^*+1}(T')$. Let $\Phi(T') := \Phi^{m^*}(T')$; we refer to $\Phi(T')$ as the set of *revealed dominated observations* (or simply *dominated observations*) in T'. Since $B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi(T'))^{\downarrow}$ cannot contain $\hat{x}$, we obtain $\hat{t} \notin \Phi(T')$. Thus $\Phi(T')$ is a strict subset of T'.

Definition 2. $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t \in T}$ *satisfies* the never-covered property under $\trianglerighteq$ *if, for any nonempty $T' \subseteq T$, the set of revealed dominated observations $\Phi(T')$ satisfies $\Phi(T') \neq T'$.*

We have shown that the never-covered property under $\trianglerighteq$ is a necessary condition for a data set to be $\trianglerighteq$-rationalizable. The main result of this paper, Theorem 1, shows that it is also sufficient. The example below illustrates how we can use the never-covered property under $\trianglerighteq$ to test the $\trianglerighteq$-rationalizability of a set of menu preference pairs.

Example 1. Consider the classical model of consumer demand with two goods. We take the preorder to be the coordinate-wise ordering $\geq$ on $X = \mathbb{R}^2_+$. Figure 1(a) depicts linear budget sets, K^p, K^q, and K^r. Suppose that $\mathcal{M}$ consists of two observations, (K^p, K^q) and (K^q, K^r), where both relations are weak. We claim that $\mathcal{M}$ is not $\geq$-rationalizable.

Suppose to the contrary that $\mathcal{M}$ is $\geq$-rationalizable, say, by a preference relation $\succsim$. Then, there exists at least one bundle $\hat{x}$ contained in K^p such that $\hat{x} \succsim K^q \cup K^r$. But this is impossible since $K^p \subseteq (K^q \cup K^r)^o$ and so there is $y \in K^q \cup K^r$ such that $y > \hat{x}$, which ensures that $y \succ \hat{x}$.[9] Notice that our argument corresponds precisely to a violation of the never-covered property under $\geq$ for $T' = T$. Since $B(T)^{\downarrow\downarrow} \bigcup B(T \cap S)^{\downarrow} = (K^q \cup K^r)^o$ covers K^p, $\Phi^1(T)$ contains the observation (K^p, K^q). Since $K^q \subseteq B(\Phi^1(T))$, $K^q \subseteq B(T)^{\downarrow\downarrow} \bigcup B((T \cap S) \cup \Phi^1(T))^{\downarrow}$ and $\Phi^2(T) = T$. Thus, $\Phi(T) = T$.

[9] For any set K, we use K^o to denote its interior.

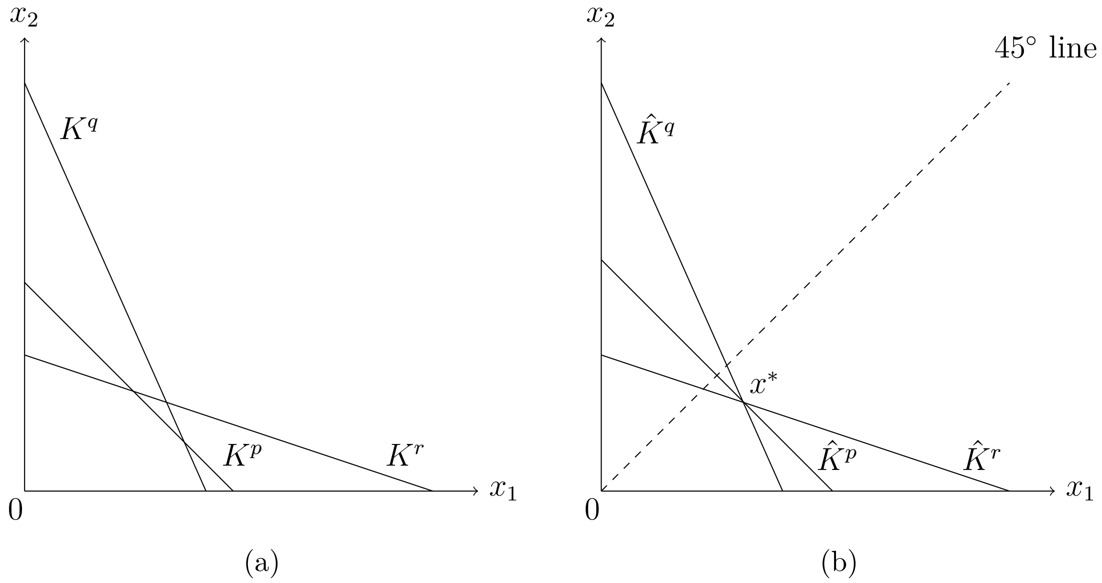

Figure 1: (a) The data set $\mathcal{M} = \{(K^p, K^q)), (K^q, K^r)\}$, where both relations are weak, is not $\geq$-rationalizable; (b) the data set $\hat{\mathcal{M}} = \{(\hat{K}^p, \hat{K}^q)), (\hat{K}^q, \hat{K}^r)\}$, where both relations are weak, is $\geq$-rationalizable.

On the other hand, the data set $\hat{\mathcal{M}} = \{(\hat{K}^p, \hat{K}^q)), (\hat{K}^q, \hat{K}^r)\}$, where both relations are weak, is $\geq$-rationalizable if the budget sets are the ones depicted in Figure 1(b).[10] In this case, the optimal bundle in each set must be x^*, and it is easy to check that the set $\Phi(T)$ is empty.

3.2 The basic result

Our proof of the sufficiency of the never-covered property under $\trianglerighteq$ in guaranteeing the $\trianglerighteq$-rationalizability of $\mathcal{M}$ proceeds by explicitly providing a way of selecting x^t in A^t for each t such that there exists a preference $\succsim$ on X that extends $\trianglerighteq$ and satisfies (1) $x^t \succsim B^t$ for all $t \in T$ and (2) $x^t \succ B^t$ for $t \in S$ (see Definition 1). Suppose that we have selected x^t from A^t for each t in some way. How do we check whether there exists a preference with the required conditions? This can be characterized by a no-cycling property which we now explain.

Let $Y = \{x^t\}_{t\in T}$. For x^t and $x^{t'}$ in Y, we say that x^t is *revealed preferred to* $x^{t'}$ and denote it by $x^t \, R \, x^{t'}$ if $x^{t'} \in B^{t\downarrow}$, and we say that x^t is *revealed strictly preferred to* $x^{t'}$ and denote it by $x^t \, P \, x^{t'}$ if either (i) $x^{t'} \in B^{t\downarrow\downarrow}$ or (ii) $t \in S$ and $x^{t'} \in B^{t\downarrow}$. The following is a no-cycling condition on the binary relations R and P.

Definition 3. *Given a data set* $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$, *a selection* x^t *from* A^t *for each* $t \in T$ *is a* no-cycling selection under $\trianglerighteq$ *if the revealed preference relations* R *and*

[10] The 45 degree line in Figure 1(b) will be used in Example 4.

P obey the following no-cycling property: there does not exist x^{t_1}, $x^{t_2} \ldots, x^{t_n}$ in $\{x^t\}_{t \in T}$ such that

$$x^{t_1} \, R \, x^{t_2} \, R \cdots R \, x^{t_n} \text{ and } x^{t_n} \, P \, x^{t_1}. \tag{1}$$

To see why this could be a plausible characterization, note that it is plainly a *necessary* condition. Indeed, suppose the preference $\succsim$ extends $\trianglerighteq$ and, with this preference, x^t satisfies (1) $x^t \succsim B^t$ for all $t \in T$ and (2) $x^t \succ B^t$ for $t \in S$. If x^t is revealed preferred to $x^{t'}$, then by definition, $x^t \succsim y \trianglerighteq x^{t'}$ for some $y \in B^t$; since $\succsim$ extends $\trianglerighteq$ and $\succsim$ is transitive, we obtain $x^t \succsim x^{t'}$. If x^t is revealed strictly preferred to $x^{t'}$, then we have either (i) $x^t \succsim y \triangleright x^{t'}$ for some $y \in B^t$ or (ii) $t \in S$ and $x^t \succ y \trianglerighteq x^{t'}$ for some $y \in B^t$; in both cases, we conclude that $x^t \succ x^{t'}$. Since $\succsim$ is transitive, we plainly cannot have x^{t_1}, $x^{t_2} \ldots, x^{t_n}$ in $\{x^t\}_{t \in T}$ satisfying (1).

We have just shown that if a data set of menu preference pairs is $\trianglerighteq$-rationalizable, then it admits a no-cycling selection under $\trianglerighteq$. Theorem 1 below states that the converse is also true and that both are equivalent to the never-covered property under $\trianglerighteq$.

Theorem 1. *Given a set of menu preference pairs $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t \in T}$ and a preorder $\trianglerighteq$, the following statements are equivalent:*

(1) *$\mathcal{M}$ is $\trianglerighteq$-rationalizable.*

(2) *$\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$.*

(3) *$\mathcal{M}$ admits a no-cycling selection under $\trianglerighteq$.*

3.3 Algorithm

Given a subset T', it is straightforward to check whether $\Phi(T') = T'$. Thus, Theorem 1 provides us with a way of checking if a set of menu preference pairs is $\trianglerighteq$-rationalizable: we need to check whether $\Phi(T') \neq T'$ for all $T' \subseteq T$. This may not seem promising as an empirical procedure, since for a data set with n observations, we would have to go through all $2^n - 1$ nonempty subsets of T to guarantee the $\trianglerighteq$-rationalizability of that set. In this subsection, we provide a simple algorithm to check whether the never-covered property under $\trianglerighteq$ holds. This algorithm requires us to check whether $\Phi(T') \neq T'$ for at most n subsets of T. Thus, the never-covered property under $\trianglerighteq$ can be checked in an efficient manner. In the Online Appendix we provide an implementation of this algorithm on experimental data, for two different preorders $\trianglerighteq$; more details are provided in Section 5.3 (Example 6).

Following the convention in the computer science literature, we use k' to denote the updated value of a variable k.

Algorithm I. Set $T^0 := T$. Set $k := 1$.

START. Derive $T^k := \Phi(T^{k-1})$. Consider the following three mutually exclusive cases:

(a). $T^k = \emptyset$.

Stop and output $\trianglerighteq$-*Rationalizable.*

(b). $\emptyset \neq T^k \subsetneq T^{k-1}$.

Go to START with $k' = k + 1$.

(c). $\emptyset \neq T^k = T^{k-1}$.

Stop and output *Not* $\trianglerighteq$-*Rationalizable.*

Note that Algorithm I is effectively checking whether $\Phi(T^k) = T^k$ for an endogenous sequence of subsets of T. We emphasize that, for a data set with n observations, Algorithm I necessarily terminates within n steps, and we only need to check at most n subsets of T.

Proposition 1 below provides the justification for Algorithm I.

Proposition 1. *The set of menu preference pairs* $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t \in T}$ *is* $\trianglerighteq$-*rationalizable if and only if Algorithm I outputs* $\trianglerighteq$-Rationalizable.

3.4 Nice rationalization when $\trianglerighteq$ is trivial

The analysis in the previous subsections holds for any preorder $\trianglerighteq$ and any partition $\{W, S\}$ of T. In this subsection, we focus on the important special case in which the rationalizing preference is not required to extend any given preorder or, put another way, the preorder $\trianglerighteq$ is simply the trivial preorder where $x \trianglerighteq y$ if and only if $x = y$.

Consider a set of menu preference pairs $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t \in T}$ where S is nonempty.[11] Since $\trianglerighteq$ is trivial, $A^{\downarrow} = A$ and $A^{\downarrow\downarrow} = \emptyset$ for all A. Thus, the procedure of iterated exclusion of dominated observations reduces to the following: for any nonempty $T' \subseteq T$,

$$\Phi^1(T') = \left\{t \in T' : A^t \subseteq B(T' \cap S)\right\} \text{ and}$$
$$\Phi^{m+1}(T') = \left\{t \in T' : A^t \subseteq B\left((T' \cap S) \cup \Phi^m(T')\right)\right\}, \text{ for } m = 1, 2, \ldots.$$

This iteration must stop at some point, i.e., there is m^* such that $\Phi^{m^*}(T') = \Phi^{m^*+1}(T')$. The set of dominated observations is $\Phi(T') := \Phi^{m^*}(T')$. By definition,

[11] If $\trianglerighteq$ is trivial, then any data set such that $S = \emptyset$ is rationalizable (by the preference relation that the agent is indifferent among all alternatives).

$\mathcal{M}$ satisfies the never-covered property under the trivial preorder if $\Phi(T') \neq T'$ for all nonempty $T' \subseteq T$. For the sake of brevity, we would simply refer to $\mathcal{M}$ as *satisfying the never-covered property*, if it satisfies the never-covered property under the trivial preorder.

The following example is a simple illustration of the application of Theorem 1 and Algorithm I to this setting.

Example 2. Let $X = \{x, y, z, r, w\}$ and suppose that $\trianglerighteq$ is trivial. The data set $\mathcal{M}$ consists of the following observations:

$$\begin{aligned} A^1 &= \{x, y\},\ B^1 = \{z, r, w\}; \\ A^2 &= \{y, z\},\ B^2 = \{x, r, w\}; \\ A^3 &= \{x, r\},\ B^3 = \{y, z, w\};\ \textit{and} \\ A^4 &= \{r, w\},\ B^4 = \{z\}. \end{aligned}$$

where A^1 is strictly preferred to B^1 and the other relations are weak. In this case, $T = \{1, 2, 3, 4\}$ and $S = \{1\}$. Since $A^4 \subseteq B^1$, we obtain $\Phi^1(T) = \{4\}$. Since $B^1 \cup B^4 = B^1$, $\Phi^2(T) = \{4\}$, and thus $\Phi(T) = \{4\}$. Algorithm I now directs us to calculate $\Phi(\{4\})$; this set is empty and so Algorithm I concludes that $\mathcal{M}$ is rationalizable. And indeed it is: for example, the preference $x \sim y \succ r \sim w \sim z$ rationalizes $\mathcal{M}$.

The following definition imposes a stronger notion of rationalizability than the one provided in Definition 1 because the preference $\succsim$ is required to have an optimum in every menu in the list of observations.

Definition 4. *A set of menu preference pairs $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t \in T}$ is* nicely rationalized *by a preference $\succsim$ on X if for all $t \in T$,* $\max(A^t; \succsim)$ *and* $\max(B^t; \succsim)$ *are nonempty, and for any $x^t \in \max(A^t; \succsim)$ and $y^t \in \max(B^t; \succsim)$, we have* (1) $x^t \succsim y^t$ *and* (2) $x^t \succ y^t$ *if $t \in S$. In this case, we say that $\mathcal{M}$ is* nicely rationalizable. *A preference $\succsim$* nicely $\trianglerighteq$-rationalizes *$\mathcal{M}$ if $\succsim$ nicely rationalizes $\mathcal{M}$ and extends $\trianglerighteq$; in this case, we say that $\mathcal{M}$ is* nicely $\trianglerighteq$-rationalizable.

In general, it is possible for a preference to rationalize a data set without it being a nice rationalization. Of course, this cannot happen when X is finite, since the existence of an optimum given a preference is then guaranteed. The next result says that it cannot happen when the preorder is trivial either, in the sense that every data set that is rationalizable (by some preference) is also nicely rationalizable (by a possibly different preference).

Theorem 2. *The following statements on the data set* $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ *are equivalent:*

(1) $\mathcal{M}$ *is rationalizable.*

(2) $\mathcal{M}$ *satisfies the never-covered property.*

(3) $\mathcal{M}$ *is nicely rationalizable.*

Example 3. As an illustration of Theorem 2, consider the case in which the data set consists of just one observation: $A^1 = \{1\}$, $B^1 = (0, 1)$, with A^1 strictly preferred to B^1. Can this data set be rationalized by a preference that extends the standard total order $\geq$ on $\mathbb{R}$? Clearly, such a rationalization exists; in fact, $\geq$ is itself the unique rationalization. However, this is not a nice rationalization since $(0, 1)$ does not have an optimum according to $\geq$. On the other hand, Theorem 2 guarantees that there *is* a nice rationalization of the strict preference of A^1 over B^1 if we do not require the rationalizing preference to extend $\geq$. And indeed it does: simply let $1 \succ r$ for all $r < 1$, and for all $r,\ r' \in (0, 1)$ let $r \sim r'$.

3.5 Strict menu preferences and strict rationalization

We now turn to the case in which $\trianglerighteq$ is trivial and $T = S$, so that the rationalizability of a data set $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ reduces to the following: there exists a preference $\succsim$ on X such that for each t, there exists $x \in A^t$ with $x \succ B^t$. This case is of particular interest, as the procedure of iterated exclusion of dominated observations ends in one round and the never-covered property has a much simpler form. To wit, since $T = S$, for any nonempty $T' \subseteq T$, $B(T' \cap S) = B(T')$. Therefore, the procedure of iterated exclusion of dominated observations reduces to the following: for any nonempty $T' \subseteq T$,

$$\Phi^1(T') = \left\{t \in T' : A^t \subseteq B(T' \cap S)\right\} = \left\{t \in T' : A^t \subseteq B(T')\right\} \text{ and}$$
$$\Phi^2(T') = \left\{t \in T' : A^t \subseteq B((T' \cap S) \cup \Phi^1(T'))\right\} = \left\{t \in T' : A^t \subseteq B(T')\right\} = \Phi^1(T').$$

Therefore, the set of dominated observations is $\Phi(T') = \{t \in T' : A^t \subseteq B(T')\}$, and the never-covered property $\Phi(T') \neq T'$ holds if and only if $A(T') \not\subseteq B(T')$.

We note that this special case of our result is (in its essentials) covered by Fishburn (1976), who establishes that there is $\succsim$ such that there is $x^t \in A^t$ with $x^t \succ B^t$ for each t if and only if

$$A(T') \not\subseteq B(T') \text{ for all nonempty } T' \subseteq T. \tag{2}$$

Following Fishburn, we refer to (2) as the *partial congruence axiom*. In fact, Fishburn's result is somewhat more general because it partially covers the case in which T is infinite.[12] We confine our attention to the case where T is finite because it is the case most relevant to empirical applications and it allows us to formulate an efficient algorithm for checking the never-covered property.[13]

The following result summarizes our findings when menu preferences are strict.

Corollary 1. *For a data set* $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ *where* $T = S$*, the following statements are equivalent:*

(1) $\mathcal{M}$ *is nicely rationalizable.*

(2) $\mathcal{M}$ *is nicely rationalizable by a strict preference.*

(3) $\mathcal{M}$ *satisfies the partial congruence axiom.*

The equivalence of the first and second statements in this corollary is due to Fishburn (1976, Lemma 1). As we have explained, the partial congruence axiom and the never-covered property are equivalent when $T = S$ and thus the equivalence of the first and third statements follows from Theorem 2.

When S is a strict subset of T (so that there are some observations where rationalization only requires $x^t \in A^t$ such that $x^t \succsim B^t$ rather than $x^t \succ B^t$), the partial congruence axiom no longer characterizes rationalizable data sets and one needs to appeal to the never-covered property. Indeed, consider the data set in Example 2; is it possible for $T = S$? The answer is 'No' because $A(T) = B(T)$ and the partial congruence axiom is violated. However, if we only require the relation in the first observation to be strict (as we did in that example), then the data set *is* rationalizable because it satisfies the (weaker) never-covered property.

We now discuss the relationship between our work and the paper of de Clippel and Rozen (2021). We first describe the problem it solves using our terminology. It assumes that the preorder $\trianglerighteq$ is trivial and considers finite data sets where each observation t has the form $(\{A^t_j\}_{j\in J(t)}, x^t)$, where $\{A^t_j\}_{j\in J(t)}$ is a collection of subsets of X. It develops an algorithm that determines if $\left\{(\{A^t_j\}_{j\in J(t)}, x^t)\right\}_{t\in T}$ admits an

[12] Fishburn (1976) considers two separate cases. For the case in which T could be infinite but A^t is required to be *finite* for each t, he shows that rationalizability is characterized by the partial congruence axiom. When T is countable and A^t is allowed to be infinite, his characterization result (Theorem 3) takes a different form, but it is equivalent to the partial congruence axiom when T is finite. The case in which T is more than countable and A^t is infinite is not covered by his results (or ours).

[13] If T is infinite, then there is obviously no hope of any algorithm for checking rationalizability. Fishburn's paper (perhaps partly because of its emphasis on the case of infinite T) does not discuss algorithms for checking the partial congruence axiom.

upper contour rationalization in the following sense: there is a strict preference $\succ$ such that, at each t, there is a set in the collection $\{A_j^t\}_{j\in J(t)}$ that is contained in the upper contour of x^t, i.e., there is $A_{j(t)}^t$ in $\{A_j^t\}_{j\in J(t)}$ such that $A_{j(t)}^t \succ x$.[14]

This problem and our menu rationalization problem have different economic motivations; however, when X is finite (so that all the relevant subsets in both problems are also finite), the two problems could be thought of as equivalent in the sense that it is always possible to convert one problem into the other, which also means that any algorithm developed for one could, in principle, be used to solve the other. That said, it should be clear from the conversion procedure (outlined in the Online Appendix) that there is no general computational reason for solving either problem in this roundabout fashion, since the converted data set would typically have more observations than the original data set. Thus, the two algorithms are best understood as distinct and serving different purposes.

3.6 Related results on menu preferences

Let $\hat{\mathcal{X}} \subseteq \mathcal{X}$ be a nonempty collection of menus and let $\succsim^M$ be a preference over $\hat{\mathcal{X}}$ (which means that $\succsim^M$ is a reflexive, transitive, and complete binary relation on $\hat{\mathcal{X}}$). Abusing our terminology somewhat, we say that a preference $\succsim$ on X nicely rationalizes $\succsim^M$ if, for all $D \in \hat{\mathcal{X}}$, the set $\max(D; \succsim)$ is nonempty and, for any $x' \in \max(D'; \succsim)$ and $x'' \in \max(D''; \succsim)$, we have $x' \succsim x''$ if $D' \succsim^M D''$ and $x' \succ x''$ if $D' \succ^M D''$.

Following Tyson (2018), we say that a menu preference $\succsim^M$ over $\hat{\mathcal{X}}$ satisfies the *cover dominance condition* if for any $A, D \in \hat{\mathcal{X}}$ and $\{B_i\}_{i\in I} \subseteq \hat{\mathcal{X}}$,

$$A \succ^M B_i \text{ for each } i \in I \text{ and } D \subseteq \cup_{i\in I} B_i \Rightarrow A \succ^M D.$$

This condition can be equivalently formulated as follows (see Scapparone (2001)): for all $A \in \hat{\mathcal{X}}$, the set $A \setminus \bigcup_{B\in \mathbf{V}(A)} B$ is nonempty, where $\mathbf{V}(A) := \{B \in \hat{\mathcal{X}} : A \succ^M B\}$. It is shown in Scapparone (2001) and Tyson (2018) that a menu preference $\succsim^M$ on $\hat{\mathcal{X}}$ is nicely rationalizable if and only if it satisfies the cover dominance condition.

Furthermore, as observed in Tyson (2018), if $\hat{\mathcal{X}}$ is finite and closed under union, the cover dominance condition is equivalent to *Kreps consistency*, which requires the following: if $A \succsim^M B$ then $A \sim^M A \cup B$.[15] Thus the eqivalence of the cover

[14] $A_{j(t)}^t \succ x$ means that $y \succ x$ for all $y \in A_{j(t)}^t$. A version of their algorithm is contained in the first working paper version of their paper; see de Clippel and Rozen (2012).

[15] We denote by $\sim^M$ the equivalence relation induced from $\succsim^M$.

dominance condition and the nice rationalizability of $\succsim^M$ can be thought of as a generalization of the following result in Kreps (1979): if X is finite and $\hat{\mathcal{X}} = \mathcal{X}$ (the collection of all nonempty subsets of X) then the menu preference $\succsim^M$ is nicely rationalizable if and only if it satisfies Kreps-consistency.

In our setup, it is assumed that $\mathcal{M}$, a finite list of preference pairs between menus A^t and B^t, is observed. Its key difference with Kreps (1979), Scapparone (2001), and Tyson (2018) is that these observations need not constitute a preference over all the menus in $\hat{\mathcal{X}} = \{A^t\}_{t\in T} \cup \{B^t\}_{t\in T}$; in other words, we do not require the observer to know how A^t compares with $A^{t'}$ or with $B^{t'}$.

Provided that the collection of menus $\hat{\mathcal{X}}$ is finite, we could construct a finite set of menu preference pairs $\mathcal{M}^*$ from a preference $\succsim^M$ on $\hat{\mathcal{X}}$ in the following way:

$$(A^t, B^t) \in \mathcal{M}^* \text{ if and only if } A^t \succsim^M B^t \text{ and } t \in S \text{ if and only if } A^t \succ^M B^t.$$

Clearly, $\mathcal{M}^*$ is nicely rationalizable (in the sense of Definition 4) by a preference $\succsim$ if and only if $\succsim$ nicely rationalizes the menu preference $\succsim^M$. By Theorem 2, the nice rationalizability of $\succsim^M$ is characterized by the never-covered property on $\mathcal{M}^*$. In particular, this means that one can efficiently check if $\succsim^M$ is nicely rationalizable by implementing Algorithm I for checking the never-covered property. It also follows immediately from Theorem 2 and the results in Tyson (2018) that, when $\hat{\mathcal{X}}$ is finite, the cover dominance condition on $\succsim^M$ and the never-covered property on $\mathcal{M}^*$ are equivalent; furthermore, if $\hat{\mathcal{X}}$ is also closed under union, then each of these conditions is equivalent to Kreps-consistency.

3.7 Continuous rationalizability

When the space of alternatives X is infinite, it is helpful to endow it with a topology and study continuous preferences. This guarantees (among other things) that the preference generates an optimum choice on compact menus and that the optimum varies continuously with the menu. For example, a continuous preference on the consumption space $\mathbb{R}^n_+$ would guarantee that the demand correspondence is nonempty when prices are strictly positive (so that the budget set is compact) and varies continuously with prices.[16]

We say that a set of menu preference pairs $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ is $\trianglerighteq$-rationalized by a continuous utility function $u : X \to \mathbb{R}$ if u represents a preference $\succsim$ that

[16] More generally, if we endow the collection of compact menus with the Hausdorff metric, then the correspondence mapping a compact menu to its optima is well-defined and upper hemicontinous.

$\trianglerighteq$-rationalizes $\mathcal{M}$ in the sense of Definition 1. Theorem 3 below provides conditions under which $\mathcal{M}$ can be rationalized by a continuous utility function.

Theorem 3. *Suppose that X is a locally compact and separable metric space and $\trianglerighteq$ is a continuous preorder on X.*[17] *For the set of menu preference pairs $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ where B^t is compact for each $t \in T$, the following statements are equivalent:*

(1) *$\mathcal{M}$ is $\trianglerighteq$-rationalizable.*

(2) *$\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$.*

(3) *$\mathcal{M}$ is $\trianglerighteq$-rationalized by a continuous utility function u.*

Note that this theorem does not assume that A^t is a compact set. It does assume that B^t is a compact set, which guarantees that for any continuous utility function u, the set $\arg\max_{x\in B^t} u(x)$ is nonempty. If, in addition, A^t is a compact set for all t, then $\arg\max_{x\in A^t} u(x)$ is also nonempty for all t and thus $\mathcal{M}$ is *nicely* $\trianglerighteq$-rationalized by a continuous utility function u if $\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$.

The following example illustrates the application of Theorem 3.

Example 4. Let $X = \mathbb{R}^n_+$ be the consumption space with n goods and let the product order $\geq$ to be underlying preorder. Then utility function u extends $\geq$ if and only if it is *strictly increasing*, in the sense that $u(x') > u(x)$ whenever $x' > x$. The order $\geq$ is continuous in the Euclidean topology on $\mathbb{R}^n_+$. Suppose that $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$, where A^t and B^t are compact sets; Theorem 3 guarantees that $\mathcal{M}$ can be nicely rationalized by a strictly increasing and continuous utility function if and only if it obeys the never-covered property under $\geq$.

There are other preorders besides the product order that could be natural in this setting. For example, $X = \mathbb{R}^n_+$ could be the space of contingent consumption, where the probability of each state is known (or part of the hypothesis). Based on these probabilities, different bundles in X could be ranked according to the first order stochastic dominance, i.e., $x \geq_{FSD} y$ if x first order stochastically dominates y. For example, suppose that the states are equiprobable; then $x \geq_{FSD} y$ and $y \geq_{FSD} x$ if the entries in y are a permutation of those in x. In this case, a utility function that extends $\geq_{FSD}$ is simply a utility function that is strictly increasing in $\geq$ and symmetric.

Obviously, $\geq_{FSD}$ is a finer order than $\geq$ in the sense that $\geq \subseteq \geq_{FSD}$. It is also straightforward to check that $\geq_{FSD}$ is a continuous preorder. Suppose that

[17] Terminology: a preorder $\trianglerighteq$ is *continuous* (or *closed*) if $\{(x, y) \in X \times X : x \trianglerighteq y\}$ is a closed set in $X \times X$.

$\mathcal{M} = \left\{(A^t, B^t)\right\}_{t \in T}$ and A^t and B^t are compact for all t; by Theorem 3, $\mathcal{M}$ satisfies the never-covered property under $\geq_{FSD}$ if and only if it admits a nice rationalization by a continuous utility function that extends $\geq_{FSD}$.

Consider the example depicted in Figure 1(b), where $\hat{K}^p$ is weakly preferred to $\hat{K}^q$ in observation 1 and $\hat{K}^q$ is weakly preferred to $\hat{K}^r$ in observation 2. Since the never-covered property under $\geq$ is satisfied, these observations can be rationalized by a continuous and strictly increasing utility function. However, they are not rationalizable by a preference that extends $\geq_{FSD}$ when states 1 and 2 are equiprobable. Notice that for every $(a, b) \in \hat{K}^p$, there is $(a', b') \in \hat{K}^q$ such that either $(a', b') > (a, b)$ or $(b', a') > (a, b)$; thus $\hat{K}^p \subseteq \hat{K}^{q\downarrow\downarrow}$.[18] Thus, the never-covered property under $\geq_{FSD}$ is violated for $\{(\hat{K}^p, \hat{K}^q)\}$.

4 Application: Revealed price preference

One of the major themes in classical consumer theory is the recovery of the utility function from indirect utility. Formally, the question can be posed in the following way. Let $v : \mathbb{R}^n_{++} \to \mathbb{R}$ be a function. What necessary and sufficient conditions on v guarantee that

$$v(p) = \max\{u(x) : p \cdot x \leq 1\} \tag{3}$$

for some function $u : \mathbb{R}^n_{+} \to \mathbb{R}$ (interpreted as the consumer's utility function)? This question has been thoroughly studied (see, for example, Krishna and Sonnenschein (1990) and Jackson (1986)) and it is well-known that the distinctive property that v necessarily satisfies is quasiconvexity.

Our objective in this section is to address a finite analog of this question: instead of recovering u from the function v we ask what conditions would allow us to recover a preference on the underlying bundles that are consistent with a *finite list* of preferences over prices. Of course, the quick and short answer to the issue before us is the never-covered property, but the additional structure of the consumer problem, with linear budget sets in Euclidean space, allows us to say more.

We work with a data set with T observations, where at each observation t, the consumer reports either a weak or strong preference between two price vectors. Following our convention, if $t \in W$, then the consumer weakly prefers the price vector p^t to q^t. If $t \in S$, then the consumer strictly prefers the price vector p^t to q^t. Without loss of generality, we normalize the income of the consumer to be 1, so that

[18] From Figure 1(b), it is clear that $\hat{K}^p$ is contained in the interior of $\hat{K}^q \cup (\hat{K}^q)'$, where $(\hat{K}^q)'$ is the reflection of $\hat{K}^q$ on the 45 degree line.

the consumer's budget set at price $p \in \mathbb{R}^n_{++}$ is

$$L(p) := \Big\{x \in \mathbb{R}^n_+ : p \cdot x \leq 1\Big\}.$$

A preference for p^t over q^t means a preference for the budget $L(p^t)$ over $L(q^t)$. Thus the set of preferences over budget sets may be denoted by $\mathcal{M} = \Big\{(L(p^t), L(q^t))\Big\}_{t \in T}$. The next result is an application of Theorem 3 to this environment.

Theorem 4. *The following statements on* $\mathcal{M} = \Big\{(L(p^t), L(q^t))\Big\}_{t \in T}$ *are equivalent:*

(1) $\mathcal{M}$ *can be rationalized by a locally nonsatiated preference on* $\mathbb{R}^n_+$.[19]

(2) $\mathcal{M}$ *satisfies the never-covered property under the product order* $\geq$.

(3) $\mathcal{M}$ *can be nicely rationalized by a strictly increasing, continuous, and concave utility function* $u : \mathbb{R}^n_+ \to \mathbb{R}$.

A straightforward application of Theorem 3 tells us that $\mathcal{M}$ satisfies the never-covered property under $\geq$ if and only if it can be rationalized by a strictly increasing and continuous utility function. The latter statement is replaced in Theorem 4 by both a weaker statement (rationalization by a locally nonsatiated preference) and a stronger statement (rationalization by a strictly increasing, continuous, and *concave* utility function). A proof of Theorem 4 is in the Appendix, but it is worth noting the following here. In establishing that Statement (1) implies (2), we cannot simply appeal to the argument in Section 3.1 because in that case we make the assumption that the rationalizing preference $\succsim$ extends a preorder; however, the local nonsatiation assumption on $\succsim$ in this context allows us to retrace that argument, essentially because for any linear budget set L (assuming strictly positive prices), $L^{\downarrow} = L$ and $L^{\downarrow\downarrow}$ is the interior of L. As for the implication from (2) to (3), the linearity of the budget sets is crucial in guaranteeing that the rationalizing utility function can be chosen to be concave; our proof of that implication combines Theorem 3 with Afriat's Theorem (see Afriat (1967)) which guarantees rationalization with a concave utility function.

We know from standard consumer theory that quasiconvexity plays a crucial role in the characterization of the indirect utility function v. The never-covered property could be thought of as the finite analog to quasiconvexity. In the case in which all the price preferences in $\mathcal{M} = \Big\{(L(p^t), L(q^t))\Big\}_{t \in T}$ are strict, i.e., $T = S$, this connection is especially clear and is presented in Corollary 2 below.

[19] A preference $\succsim$ is *locally nonsatiated* if for every $x \in \mathbb{R}^n_+$ and every open neighborhood N around x, there exists $x' \in N$ such that $x' \succ x$.

To motivate the characterizing condition in Corollary 2, suppose that $\mathcal{M}$ is rationalized by a continuous utility function u. This induces an indirect utility function v (as defined by (3)) that is decreasing in prices and quasiconvex. Given $T' \subseteq T$, there is $t^* \in T'$ such that $v(p^{t^*}) \geq v(p^t)$ for all $t \in T'$ and $s^* \in T'$ such that $v(q^{s^*}) \geq v(q^t)$ for all $t \in T'$. Since $T = S$, $v(p^{s^*}) > v(q^{s^*})$ and since v is quasiconvex, $v(q^{s^*}) \geq v(q)$ for all $q \in \operatorname{conv}\Big(\{q^s\}_{s \in T'}\Big)$ (the convex hull of $\{q^s\}_{s \in T'}$). Thus

$$v(p^{t^*}) \geq v(p^{s^*}) > v(q^{s^*}) \geq v(q)$$

for all $q \in \operatorname{conv}\Big(\{q^s\}_{s \in T'}\Big)$. Since v is decreasing, we conclude that $p^{t^*} \not\geq q$ for any $q \in \operatorname{conv}\Big(\{q^s\}_{s \in T'}\Big)$. It turns out that this quasiconvex-like property is precisely equivalent to the never-covered property.

Corollary 2. *When $T = S$, the following statements on $\mathcal{M} = \Big\{(L(p^t), L(q^t))\Big\}_{t \in T}$ are equivalent:*

(1) *$\mathcal{M}$ can be rationalized by a preference on $\mathbb{R}^n_+$.*

(2) *$\mathcal{M}$ has the following property:*

> *for any nonempty $T' \subseteq T$, there is $t^* \in T'$ such that $p^{t^*} \not\geq q$ for any $q \in \operatorname{conv}\Big(\{q^s\}_{s \in T'}\Big)$.*

(3) *$\mathcal{M}$ can be nicely rationalized by a strictly increasing, continuous, and concave utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$.*

We provide an extension of Corollary 2 to the case where W is nonempty in Section A of the Online Appendix.

5 Application: Coarse rationalizability

So far in this paper, we have considered the rationalization of a set of menu preference pairs. In this section, we discuss a formally related but economically distinct issue, namely, the *rationalization of choices from menus*. Our contribution is to provide a method for testing rationalizability in situations where observations are *coarse*, in a sense we shall make specific. Among other things, we provide an extension of Afriat's Theorem to this environment. In turn, an application of this generalized Afriat's Theorem provides us with a way to compute the size of the errors needed to rationalize a data set where choices are made, or observed, with error.

5.1 Four concepts of rationalization

Suppose that at observation t, there is a menu C^t and a set $A^t \subseteq C^t$. Two notions of rationalization are commonly used in analyses of this type. The **first concept** requires a preference $\succsim$ such that $A^t = \max(C^t; \succsim)$ for all $t \in T$; Richter's Theorem (see Richter (1966)) characterizes data sets which are rationalizable in this sense. The **second concept** requires a preference $\succsim$ such that $A^t \subseteq \max(C^t; \succsim)$ for all $t \in T$; Afriat's Theorem (and its generalizations to nonlinear domains) characterize data sets that satisfy this concept of rationalization. Loosely speaking, the first notion of rationalization is the one most commonly used in the theoretical revealed preference literature; on the other hand, empirical work using revealed preference have mostly relied on the second (weaker) notion, which is unsurprising since it does not posit that the observer has observed all the optimal choices, but only one, or some, of them.

A **third concept** of rationalization has been characterized in Fishburn (1976), where the set of optimal points is required to be contained in A^t; in other words, $\max(C^t; \succsim) \subseteq A^t$. Obviously, Fishburn's concept generalizes the one in Richter's Theorem by allowing some elements of A^t to be nonoptimal, but it retains the requirement that nothing outside of A^t is optimal. This suggests that a **fourth concept** of rationalization may be useful in empirical applications: one that allows for the possibility that some elements in A^t are nonoptimal (following Fishburn) and also that some elements outside of A^t are optimal (following Afriat). In formal terms, it requires that $\max(C^t; \succsim) \cap A^t \neq \emptyset$.

The revealed preference literature since the 1970s have by and large neglected Fishburn's rationalization concept. We think that Fishburn's concept, as well as the relaxation of that concept which we just proposed, deserves notice because they are relevant to empirical applications of revealed preference. These concepts are applicable whenever there are *coarse observations*, where the observer knows (or hypothesizes) that there is an optimal choice found in A^t, but is agnostic about precisely which alternatives within A^t are optimal. There are at least three broad scenarios where it is useful to think of coarse observations.

(1) The most obvious cases are those where the observations are simply known to be imprecise. For example, a researcher may have information on how much is spent on broad categories of goods, without knowing the allocation within each category. Alternatively, a researcher may have records on a consumer's credit card purchases, which puts a *lower* bound on how money is spent each month on different goods, but does not provide the precise breakdown of monthly expenditure since there could be

goods bought with cash.

(2) There could be situations where some alternative y^t is recorded as the choice from C^t but, in testing for rationality or estimating the preference, the researcher may wish to accommodate the possibility that choices were observed with error; this could be accomplished by defining a neighborhood A^t around y^t (in some sense appropriate to the specific context) and then checking if there is a preference $\succsim$ with $\max(C^t; \succsim) \cap A^t \neq \emptyset$ for all $t \in T$.

(3) In experimental settings, it is common to find subjects whose choice behavior are not exactly consistent with rationality. Since the choices y^t are typically observed perfectly, it is implausible to attribute the rationality violations to observational errors. Nonetheless, one could still use the size of the neighborhood A^t (suitably measured) as a way of comparing the rationality of different experimental subjects; those who require large A^ts to rationalize their behavior can be deemed less rational than those where A^t is just a small neighborhood of y^t.

5.2 Coarse data sets and menu preferences

We consider an observer who has a finite set of coarse observations of an agent's choices. We denote a *coarse data set* by $\mathcal{O} = \left\{(A^t, C^t)\right\}_{t \in T}$ where $\{W, S\}$ is a partition of T and for each $t \in T$ we have $\emptyset \neq A^t \subseteq C^t$. The interpretation is as follows. When $t \in W$, A^t contains at least one choice of the agent in C^t; when $t \in S$, A^t contains all the choices of the agent in C^t. The observer would like to recover a preference $\succsim$ that rationalizes the data in the following sense.

Definition 5. *A preference $\succsim$ on X* rationalizes *the coarse data set $\mathcal{O} = \left\{(A^t, C^t)\right\}_{t \in T}$ if*

(1) $\max(C^t; \succsim) \cap A^t \neq \emptyset$ *for each $t \in W$, and*

(2) $\max(C^t; \succsim) \subseteq A^t$ *for each $t \in S$.*

If $\succsim$ exists, we say that $\mathcal{O}$ is rationalizable. *If $\succsim$ can be chosen to rationalize $\mathcal{O}$ and extend a given preorder $\trianglerighteq$, then $\mathcal{O}$ is* $\trianglerighteq$-rationalizable.

Obviously, conditions (1) and (2) in this definition correspond precisely to the fourth and third concepts of rationalization discussed in Section 5.1. Note that if $T = W$, then every coarse data set is trivially rationalized by a preference that is indifferent across all alternatives; in this case, the rationalizability problem is interesting only if the preference is required to be locally nonsatiated or to extend some preorder $\trianglerighteq$.

Checking if a coarse data set is rationalizable is straightforward, given the results on menu preference pairs we have developed in Section 3. Indeed, for any coarse data set $\mathcal{O} = \left\{(A^t, C^t)\right\}_{t \in T}$, we could construct the following set $\mathcal{M}^*$ of menu preference pairs: the menu A^t is weakly preferred to the menu C^t for all $t \in T$ and A^t is strictly preferred to the menu $C^t \setminus A^t$ for $t \in S$. Clearly, $\mathcal{M}^*$ is rationalized by a preference $\succsim$ if and only if the coarse data set $\mathcal{O}$ is rationalizable by the same preference $\succsim$. Thus every result we have on the rationalizability (or $\trianglerighteq$-rationalizability) of the set of menu preference pairs $\mathcal{M}^*$ has an analog for $\mathcal{O}$.

In the following subsection, we extend Afriat's Theorem to coarse data sets.

5.3 A generalization of Afriat's theorem

We consider a data set $\mathcal{O} = \left\{(A^t, L(p^t, y^t))\right\}_{t \in T}$ where for each $t \in T$, $p^t \in \mathbb{R}^n_{++}$ is the price vector, y^t is the total expenditure, and $L(p^t, y^t) := \{x \in \mathbb{R}^n_+ : p^t \cdot x \leq y^t\}$ is the budget set at observation t.[20] Departing from the standard setting of Afriat's Theorem, the observer does not know the exact choice of the consumer and only knows that the choice lies in $A^t \subseteq L(p^t)$. The following result provides us with a test of coarse rationalizability in this setting.

Theorem 5. *Let $\mathcal{O} = \left\{(A^t, L(p^t, y^t))\right\}_{t \in T}$ be a coarse data set where $T = W$ and $A^t \subseteq L(p^t, y^t)$ for all $t \in T$. The following statements are equivalent:*

(1) *$\mathcal{O}$ can be rationalized by a locally nonsatiated preference.*

(2) *$\mathcal{O}$ satisfies the never-covered property under the product order $\geq$.*[21]

(3) *$\mathcal{O}$ can be rationalized by a strictly increasing, continuous, and concave utility function.*

Example 5. In studies of consumer demand, a researcher would often not have information on the demand for every relevant good. A common way to address this issue is to perform some aggregation procedure across goods, even though this approach is strictly valid only under stringent conditions on the utility function and/or the pattern of prices changes.

To be more specific, suppose that at observation t, the information available consists of the prices of all goods $p^t \in \mathbb{R}^n_{++}$, the demand for the first $m - 1$ goods, and the total expenditure on the remaining goods (which we denote by $c^t_{m,n}$). In

[20] Note that we depart from the convention and notation of the previous section by *not* normalizing expenditure at 1. This presentation is more appropriate in this section to highlight the fact that total expenditure y^t is part of the observer's data.

[21] In this statement, we are interpreting $\mathcal{O}$ as a set of weak menu preference pairs.

other words, the actual demand for goods m, $m+1,\ldots,n$ is not observed. To get round this problem, the researcher could construct a price index for those goods, $\bar{p}^t_m$, which would be a function of their prices $(p^t_m, p^t_{m+1},\ldots,p^t_n)$, with the corresponding demand for the composite good being $\bar{x}^t_m = c^t_{m,n}/\bar{p}^t_m$. In this way, the researcher creates a data set of the standard form, with prices $(p^t_1, p^t_2,\ldots,\bar{p}^t_m)$ and demand $(x^t_1, x^t_2,\ldots,\bar{x}^t_m)$ for m goods at each observation.

Coarse data sets offer a potentially useful alternative approach to tackle this problem. At observation t, the researcher observes x^t_i for $i=1,\ldots,m-1$ and $c^t_{m,n}$. Thus the demand of the consumer must lie in the set

$$A^t = \{x \in \mathbb{R}^n_+ : x_i = x^t_i \text{ for } i=1,\ldots,m-1 \text{ and } \textstyle\sum_{i=m}^n p_i x_i = c^t_{m,n}\}.$$

The corresponding coarse data set is $\mathcal{O} = \{(A^t, L(p^t,y^t))\}_{t\in T}$, where $y^t = \sum_{i=1}^{m-1} p^t_i x^t_i + c^t_{m,n}$. This can be analyzed using Theorem 5.

As an illustration, suppose that $\mathcal{O}$ consists of two observations where

$$\begin{array}{llll} p^1 = (2, 2.5, 3.5), & x^1_1 = 1.5, & c^1_{2,3} = 9, & y^1 = 12 \\ p^2 = (4,3,3), & x^1_1 = 3, & c^1_{2,3} = 4.5, & y^2 = 16.5. \end{array}$$

This data set is coarse rationalizable. Indeed, the bundle $\tilde{x} = (1.5, 9/2.5, 0)$ is in A^1 but $p^2 \cdot \tilde{x} = 16.8 > 16.5$, so it is not in $L(p^2,y^2)$. Given that there are just two observations, this is enough to guarantee that $\mathcal{O}$ satisfies the never-covered property under $\geq$.

On the other hand, suppose we were to aggregate goods 2 and 3 into a composite commodity, with the price of the composite being 3 (the average price of its constituent goods) at both observations 1 and 2. Then the demand for the composite good at these observations are $\bar{x}^1_2 = 9/3 = 3$ and $\bar{x}^2_2 = 4.5/3 = 1.5$. The corresponding two-good data set has

$$\begin{array}{lll} p^1 = (2,3), & x^1 = (1.5,3), & y^1 = 12; \\ p^2 = (4,3), & x^2 = (3,1.5) & y^2 = 16.5. \end{array}$$

It is straightforward to check that it violates GARP and is not rationalizable.

Example 6. (The Perturbation Index) Consider a researcher who observes that the consumer chooses the bundle x^t from the budget set $L(p^t,y^t)$. It is common for data sets to be not rationalizable, in the sense that there is no locally nonsatiated preference $\succsim$ such that $x^t \succsim x$ for all $x \in L(p^t,y^t)$. As explained in Section 5.1,

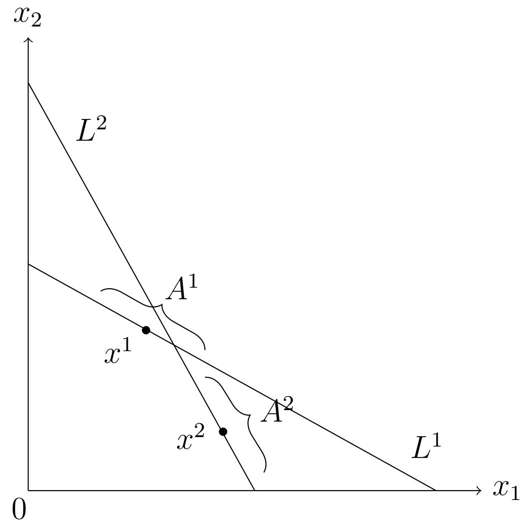

Figure 2: The data set $\{(x^1, L^1), (x^2, L^2)\}$ is not rationalizable, but $\{(A^1, L^1), (A^2, L^2)\}$ (as depicted) is rationalizable.

this can be attributed to errors committed when observing x^t or errors made by the consumer when choosing x^t. Thus it is natural to use the size of this error as a measure of how far a data set is from being rationalizable.[22] To be precise, we could allow for the optimal bundle to be in

$$A^{t,\kappa} = \{x \in L(p^t, y^t) : \ p^t \cdot x = y^t \text{ and } |p_i^t x_i - p_i^t x_i^t| \leq \kappa y^t \text{ for all } i\},$$

where $\kappa \in [0, 1]$. In other words, the optimal expenditure on good i is allowed to deviate from $p_i^t x_i^t$ but not by more than κy^t. This is illustrated in Figure 2, where the 'original' data set $\{(x^1, L^1), (x^2, L^2)\}$ is not rationalizable, but $\{(A^1, L^1), (A^2, L^2)\}$ (as depicted) is rationalizable. More generally, our extension of Afriat's Theorem provides a way to check if $\mathcal{O}^\kappa = \left\{(A^{t,\kappa}, L(p^t, y^t))\right\}_{t \in T}$ is $\geq$-rationalizable. A natural measure of the size of $\mathcal{O}$'s departure from rationalizability is then provided by the *perturbation index*[23]

$$\kappa^* := \inf \{\kappa : \mathcal{O}^\kappa \text{ is } \geq\text{-rationalizable}\}.$$

In order to see how this index behaves on real data and also to ascertain that our algorithm for checking the never-covered property actually works, we calculate

[22]This is by no means the only measure of a data set's 'distance' from rationalizability. The Online Appendix mentions prominent alternative approaches.

[23] The idea of assessing a data set's departure from rationality through the size of the errors on observed choices is also found in Varian (1985), though that paper considers a different model with a different revealed preference test; the Online Appendix explains this in greater detail.

κ^* for different subjects in the budgetary experiment carried out by Choi et al. (2007). The results are reported in the Online Appendix. (In this experiment there are two commodities, 50 budgetary observations on each subject, and 47 subjects.) Algorithm I is used to determine if $\mathcal{O}^\kappa$ satisfies the never-covered property under $\geq$, for different values of κ, in order to pin down κ^*. We also carry out a similar procedure to calculate the perturbation index in the case where the rationalizing preference is required to extend $\geq_{FSD}$ (as defined in Example 4).

6 Application: Multiple preferences

In this section, we investigate the observable restrictions of the multiple preferences model, as presented in Aizerman and Malishevski (1981), Moulin (1985), and Salant and Rubinstein (2008). In contrast with the single preference model, the choice behavior of the agent may be a result of multiple rationales. Formally, the agent has a set Π of strict preferences, and she chooses

$$F_\Pi(A) := \Big\{x : x = \max(A; \succ) \text{ for some } \succ \in \Pi\Big\} \text{ for each menu } A.$$

We represent the observed choice behavior of the agent by a data set (Σ, f), where $\Sigma \subseteq \mathcal{X}$ and $f(A)$ is the collection of alternatives that the agent chooses in $A \in \Sigma$. We say that (Σ, f) is *rationalizable by multiple preferences* (or *multi-rationalizable*, for short) if there exists a set Π of strict preferences such that

$$f(A) = F_\Pi(A) \text{ for all } A \in \Sigma.$$

Example 7 (A data set that is not rationalizable by multiple preferences)**.** Suppose $X = \{x, y, z\}$ and $\Sigma = \{\{x, y\}, \{y, z\}, \{x, y, z\}\}$. Let f be defined as follows:[24]

(1) $f(\{x, y\}) = x$;

(2) $f(\{y, z\}) = y$;

(3) $f(\{x, y, z\}) = \{x, z\}$.

To see that (Σ, f) is not multi-rationalizable, suppose to the contrary, that it is. Then there exists a set Π of strict preferences such that $f(A) = F_\Pi(A)$ for all $A \in \Sigma$. The first observation $f(\{x, y\}) = x$ reveals that $x \succ y$ for all $\succ \in \Pi$, and the second observation $f(\{y, z\}) = y$ reveals that $y \succ z$ for all $\succ \in \Pi$. By transitivity, it must be that $x \succ z$ for all $\succ \in \Pi$, which contradicts with the third observation that $f(\{x, y, z\}) = \{x, z\}$.

[24] We abuse the notation by suppressing the set delimiters, e.g., writing x rather than $\{x\}$.

In what follows, we identify two ways of characterizing those (Σ, f) that are rationalizable by multiple preferences.

Characterization 1. A straightforward application of the results in Section 3 provides us with a test of whether (Σ, f) is multi-rationalizable. For notational simplicity, we denote by $g(A) := A \setminus f(A)$ the set of alternatives in A that are not chosen. Suppose that (Σ, f) can be rationalized by a set of strict preferences Π. Then for each $A \in \Sigma$ and $x \in f(A)$, there exists a strict preference $\succ \in \Pi$ such that x is optimal in A according to $\succ$. Furthermore, for any other set $A' \in \Sigma$, the optimal element according to $\succ$ must be contained in $f(A')$. It follows that, *for each $A \in \Sigma$ and $x \in f(A)$, the set of menu preference pairs*

$$\mathcal{M}_{A,x} := \{(x, A \setminus x)\} \cup \Big\{(f(A'), g(A'))\Big\}_{A' \in \Sigma,\, A' \neq A} \tag{4}$$

(where all menu preferences are strict) must be rationalizable by a strict preference. Furthermore, this is also a sufficient condition for multi-rationalizability: indeed, if $\mathcal{M}_{A,x}$ is rationalizable by some strict preference $\succ_{A,x}$, then clearly the set $\Pi = \{\succ_{A,x} : A \in \Sigma,\ x \in f(A)\}$ would rationalize (Σ, f) with multiple preferences. Since for each A and $x \in f(A)$, we can check if $\mathcal{M}_{A,x}$ is rationalizable by a strict preference by checking if the partial congruence axiom holds (see Corollary 1), we obtain a characterization of the multi-rationalizability of (Σ, f).

Characterization 2. Clearly, if (Σ, f) is rationalizable by multiple preferences then the set of menu preference pairs $\Big\{(f(A), g(A))\Big\}_{A \in \Sigma}$ is rationalizable by a strict preference. By Corollary 1, the latter holds if and only if

$$\cup_{A \in \Sigma'} f(A) \not\subseteq \cup_{A \in \Sigma'} g(A) \text{ for all nonempty } \Sigma' \subseteq \Sigma \tag{5}$$

or, in other words, that $\Big(\cup_{A \in \Sigma'} f(A) \setminus \cup_{A \in \Sigma'} g(A)\Big)$ is nonempty for all nonempty $\Sigma' \subseteq \Sigma$. The multi-rationalizability of (Σ, f) can be characterized by a strengthening of condition (5). Specifically, we require the following:[25]

$$\text{for any nonempty } \Sigma' \subseteq \Sigma \text{ and } B \in \Sigma, \quad \Big(\cup_{A \in \Sigma'} f(A) \setminus \cup_{A \in \Sigma'} g(A)\Big) \subseteq B \Longrightarrow f(B) \cap \Big(\cup_{A \in \Sigma'} g(A)\Big) = \emptyset. \tag{6}$$

To help us understand condition (6) better, note that it is a necessary condition

[25] To see that this is a strengthening of (5), suppose that it is violated. Then there exists some nonempty $\Sigma' \subseteq \Sigma$ such that $\cup_{A \in \Sigma'} f(A) \subseteq \cup_{A \in \Sigma'} g(A)$. Fix an arbitrary nonempty set $B \in \Sigma'$. Since $\cup_{A \in \Sigma'} f(A) \setminus \cup_{A \in \Sigma'} g(A) = \emptyset \subseteq B$, it follows from (6) that $f(B) \cap \big(\cup_{A \in \Sigma'} g(A)\big) = \emptyset$. We have arrived at a contradiction, since $f(B) \subseteq \cup_{A \in \Sigma'} f(A) \subseteq \cup_{A \in \Sigma'} g(A)$.

for (Σ, f) to be rationalizable by a set of strict preferences Π. Fix an arbitrary nonempty $\Sigma' \subseteq \Sigma$ and $B \in \Sigma$. Observe that if $x \in F_\Pi(\cup_{A\in\Sigma'} A)$ then there must exist $\underline{A} \in \Sigma'$ such that $x \in \underline{A}$; furthermore, whenever this occurs, $x \in F_\Pi(\underline{A}) = f(\underline{A})$ and not in $g(\underline{A})$. It follows that

$$F_\Pi\left(\cup_{A\in\Sigma'} A\right) \subseteq \cup_{A\in\Sigma'} f(A) \setminus \cup_{A\in\Sigma'} g(A). \tag{7}$$

If $\Big(\cup_{A\in\Sigma'}\ f(A) \setminus \cup_{A\in\Sigma'} g(A)\Big) \subseteq B$, then $F_\Pi(\cup_{A\in\Sigma'} A) \subseteq B$. Since $F_\Pi(B) = f(B)$ consists of those elements in B that are optimal in B for some preference in Π,

$$f(B) \cap \Big[\left(\cup_{A\in\Sigma'} A\right) \setminus F_\Pi(\cup_{A\in\Sigma'} A)\Big] = \emptyset.$$

Furthermore (7) tells us that $F_\Pi\left(\cup_{A\in\Sigma'} A\right) \subseteq \cup_{A\in\Sigma'} A \setminus \cup_{A\in\Sigma'} g(A)$ and thus

$$\cup_{A\in\Sigma'} g(A) \subseteq \Big[\left(\cup_{A\in\Sigma'} A\right) \setminus F_\Pi(\cup_{A\in\Sigma'} A)\Big].$$

We conclude that $f(B) \cap \Big(\cup_{A\in\Sigma'}\ g(A)\Big) = \emptyset$.

The proof that (6) is also sufficient to guarantee the multi-rationalizability of (Σ, f) is more complicated and is obtained by showing that it implies the condition in Characterization 1, i.e., $\mathcal{M}_{A,x}$ is rationalizable by a strict preference for all $A \in \Sigma$ and $x \in f(A)$.[26] The following result summarizes our discussion.

Theorem 6. *The following statements on (Σ, f) are equivalent:*

(1) (Σ, f) *is rationalizable by multiple preferences.*

(2) *For each $A \in \Sigma$ and $x \in f(A)$, the set of menu preference pairs $\mathcal{M}_{A,x}$ (as defined by (4)) is rationalizable by a strict preference.*

(3) (Σ, f) *satisfies (6).*

Condition (6) does not appear to be very promising as a practical way of testing multi-rationalizability, since it involves checking the condition for all sub-collections Σ' and for all $B \in \Sigma$. In fact, that is not necessary. For each $B \in \Sigma$, we need only check (6) for a strictly nested sequence of subcollections of Σ, beginning with Σ itself. So (6) needs to be checked at most $|\Sigma|$ times for each B: for $\Sigma' = \Sigma$, some $\Sigma' = \Sigma^1 \subsetneq \Sigma$, some $\Sigma' = \Sigma^2 \subsetneq \Sigma^1$, and so on. Given that there are $|\Sigma|$ different values of B, the total number of checks of (6) does not exceed $|\Sigma|^2$. The algorithm below spells out how, for a given B, the sequence Σ^1, $\Sigma^2 \ldots$ could be obtained.

[26] Moulin (1985) uses a similar proof strategy when proving that Aizerman and Chernoff axioms characerize multi-rationalizability in the case of complete data, i.e., when $\Sigma = \mathcal{X}$ (see Corollary 3). In other words, with complete data, those axioms guarantee that $\mathcal{M}_{A,x}$ is rationalizable by a strict preference for all $A \in \Sigma$ and $x \in f(A)$.

Algorithm II. Set $\Sigma' = \Sigma$.

START. Derive $N(\Sigma', B) := (\cup_{A\in\Sigma'} f(A)) \setminus ((\cup_{A\in\Sigma'} g(A)) \cup B)$. Consider the following four mutually exclusive cases:

(a). $\Sigma' = \emptyset$.
Stop and output B *satisfies (6) for all* $\Sigma' \subseteq \Sigma$.

(b). $\Sigma' \neq \emptyset$ and $N(\Sigma', B) \neq \emptyset$.
Go to START with $\Sigma' = \{A \in \Sigma' : f(A) \cap N(\Sigma', B) = \emptyset\}$.

(c). $\Sigma' \neq \emptyset$, $N(\Sigma', B) = \emptyset$ and $f(B) \cap (\cup_{A\in\Sigma'} g(A)) = \emptyset$.
Stop and output B *satisfies (6) for all* $\Sigma' \subseteq \Sigma$.

(d). $\Sigma' \neq \emptyset$, $N(\Sigma', B) = \emptyset$ and $f(B) \cap (\cup_{A\in\Sigma'} g(A)) \neq \emptyset$.
Stop and output B *fails (6) for some* $\Sigma' \subseteq \Sigma$.

Proposition 2. *For a given* $B \in \Sigma$*, Algorithm* II *verifies if (6) holds for all* $\Sigma' \subseteq \Sigma$*.*

Theorem 6 holds for any data set (Σ, f). In the special case of complete data, that is, when $\Sigma = \mathcal{X}$, Aizerman and Malishevski (1981) shows that the multiple preferences model could be characterized by the following two axioms:

Chernoff: $A \subseteq B \Rightarrow f(B) \cap A \subseteq f(A)$ for all $A, B \in \mathcal{X}$.

Aizerman: $f(B) \subseteq A \subseteq B \Rightarrow f(A) \subseteq f(B)$ for all $A, B \in \mathcal{X}$.

In words, the Chernoff axiom says that a best choice in some set is still best if the set shrinks. The Aizerman axiom says that deleting from a given set some choices outside the choice set cannot make new choices chosen. We establish this result as a corollary of Theorem 6 by showing that condition (6) follows from the Chernoff and Aizerman axioms in the case of complete data.

Corollary 3 (Aizerman and Malishevski (1981))**.** *The data set* $(\mathcal{X}, f)$ *is rationalizable by multiple preferences if and only if it satisfies the Chernoff axiom and the Aizerman axiom.*

7 Application: Minimax regret

The minimax regret criterion was first suggested in Savage (1951) to model an agent who anticipates regret and chooses to minimize the worst-case regret that could occur. In this section, we investigate the observable restrictions of this model by applying the results of Section 3.

Let X be a finite nonempty set of alternatives. The utility of an alternative x depends on the realization of the state; we denote the state space by U, a typical element in this space by u, and the utility of the alternative x in state u by $u(x)$. The *regret* of choosing x relative to y is $u(y) - u(x)$ if the state is u and the worst-case regret of choosing x relative to y is

$$\phi_U(x, y) := \max_{u \in U} \Big\{ u(y) - u(x) \Big\}.$$

In a menu $M \subseteq X$, the worst-case regret of choosing x is thus $\max_{y \in M} \phi_U(x, y)$. The agent who uses the minimax regret decision criterion chooses those alternatives that lead to the lowest worst-case regret, i.e., at the menu M, the agent chooses

$$R_U(M) := \arg\min_{x \in M} \Big\{ \max_{y \in M} \phi_U(x, y) \Big\}.$$

As in the previous section, we represent the observed choice behavior of an agent by (Σ, f), where $\Sigma \subseteq \mathcal{X}$ and $f(M)$ is the choice of the agent in $M \in \Sigma$. We say that (Σ, f) is *rationalizable by the minimax regret model* if there exists a finite set U of utility functions such that

$$f(M) = R_U(M) \quad \text{for all } M \in \Sigma. \tag{8}$$

Notice that following Kreps (1979) and Dekel et al. (2001) (see also Dekel et al. (2007)) the set of states/utility functions U is *not* known to the observer and is something that has to be recovered as part of the rationalization of (Σ, f). This is the key difference between our result (Theorem 7 below) and those results which axiomatize the minimax regret model under the assumption that the set of states is known and the alternatives are acts mapping states to outcomes which are also known to the observer.[27]

Given the flexibility available in constructing the set of utility functions, readers might wonder whether the minimax model has *any* observable restrictions in our setup. To get this out of the way, we present a data set (Σ, f) that is not rationalizable under the minimax regret model.

[27] Milnor (1954) and Stoye (2011) axiomatize a model where the agent minimizes regret, with the regret associated with an act x in menu M given by $\max_{y \in M} \{\max_{s \in \mathcal{S}} Ev(y(s)) - Ev(x(s))\}$. (In this formulation, an act x maps each state s (in the set S) to an objective lottery $x(s)$ in an outcome space and $Ev(x(s))$ is the expected utility of this lottery (with Bernoulli index v).) This is a case of our model if we set $U = \{Ev(\cdot, s)\}_{s \in S}$. Conversely, the elements in X in our formulation could be formulated as acts by letting $S = U$, the outcome space be $\mathbb{R}$, identifying $x \in X$ with the act given by $x(u) = u(x)$, and letting the Bernoulli index be the identity function.

Example 8 (A data set that is not rationalizable under the minimax regret model)**.** Let $X = \{x, y, z, w\}$, with (Σ, f) consisting of the following observations: $f(X) = x$, $f(X \setminus z) = y$, and $f(X \setminus w) = y$.

Suppose that (Σ, f) is rationalizable by the minimax regret model. Then since $f(X) = x$ and $f(X \setminus z) = y$, it must be the case that

$$\max\{\phi_U(y, z), \phi_U(y, w), \phi_U(y, x)\} = \phi_U(y, z)$$

and, in particular, $\phi_U(y, z) > \phi_U(y, w)$. Similarly, since $f(X) = x$ and $f(X \setminus w) = y$, we obtain $\phi_U(y, w) > \phi_U(y, z)$, which is a contradiction.

Our objective is to show how we can check the rationalizability of (Σ, f) by the minimax regret model by checking the rationalizability of an appropriately constructed data set of menu preference pairs. To that end, we first make the observation that given any finite set U of utility functions defined on X,

$$R_U(M) = \arg\min_{x \in M} \Big\{ \max_{y \in M \setminus x} \phi_U(x, y) \Big\} \quad \text{for all } M \in \Sigma; \tag{9}$$

in other words, when considering the regret of choosing x from the menu M, one can omit comparing x with itself. We leave the reader to verify that this is true. Given this observation, the rationalizability of (Σ, f) by the minimax regret model is equivalent to the existence of a finite set U of utility functions such that,

for all $M \in \Sigma$,

$$\max_{y \in M \setminus x} \phi_U(x, y) > \max_{y \in M \setminus x^*} \phi_U(x^*, y) \text{ if } x \in M \setminus f(M),\ x^* \in f(M), \text{ and} \tag{10}$$

$$\max_{y \in M \setminus x^*} \phi_U(x^*, y) = \max_{y \in M \setminus x^{**}} \phi_U(x^{**}, y) \text{ if } x^*,\ x^{**} \in f(M). \tag{11}$$

Given (Σ, f), for each $M \in \Sigma$ and $x \in M$ with $x \neq f(M)$, let $\bar{A}_{x,M} = x \times (M \setminus x)$. We define the set of menu preference pairs $\bar{\mathcal{O}} = \bar{\mathcal{O}}^S \cup \bar{\mathcal{O}}^W$, where

$$\bar{\mathcal{O}}^S := \{(\bar{A}_{x,M}, \bar{A}_{x^*,M}) : x \in M \setminus f(M),\ x^* \in f(M) \text{ and } M \in \Sigma\}, \tag{12}$$

$$\bar{\mathcal{O}}^W := \{(\bar{A}_{x^*,M}, \bar{A}_{x^{**},M}) : x^*,\ x^{**} \in M \setminus f(M), \text{ and } M \in \Sigma\}. \tag{13}$$

Notice that if (Σ, f) is rationalizable by a minimax regret model, then $\bar{\mathcal{O}}$ can be rationalized by a preference on $X \times X$. This is clear since (10) and (11) guarantees that ϕ_U rationalizes $\bar{\mathcal{O}}$. Conversely, if $\bar{\mathcal{O}}$ is rationalizable, then there is a utility function $W : \bar{X} \to \mathbb{R}$, where $\bar{X} = \{(x, x') \in X \times X : x \neq x'\}$, such that[28]

$$f(M) = \arg\min_{x \in M} \left\{ \max_{y \in M \setminus x} W(x, y) \right\} \quad \text{for all } M \in \Sigma.$$

What is less obvious is that one could find a set of utility functions U, such that the preference generated by W has a representation of the form ϕ_U, but that turns out to be true.

It fact, it is possible to simplify the test by checking the rationalizability of a smaller set of menu preference pairs $\hat{\mathcal{O}} = \hat{\mathcal{O}}^S \cup \bar{\mathcal{O}}^W$ where

$$\hat{\mathcal{O}}^S = \{(\bar{A}_{x,M}, \bar{A}_{x^*,M}) : x \in (M \setminus f(M)) \cap (\cup_{D \in \Sigma} f(D)),\ x^* \in f(M) \text{ and } M \in \Sigma\}.$$

So $\hat{\mathcal{O}}$ only contains those pairs $(\bar{A}_{x,M}, \bar{B}_{x^*,M})$ such that x is chosen in some menu $D \in \Sigma$. When the number of elements in $\cup_{D \in \Sigma} D$ is large compared to $\cup_{D \in \Sigma} f(D)$, the number of observations in $\hat{\mathcal{O}}$ will be much smaller than that in $\bar{\mathcal{O}}$. In the important special case where $f(M)$ is unique for all $M \in \Sigma$, the set $\bar{\mathcal{O}}^W$ is empty and $\hat{\mathcal{O}} = \hat{\mathcal{O}}^S$ will have no more than $|\Sigma|^2$ observations.

Theorem 7. (Σ, f) *is rationalizable by the minimax regret model if and only if the corresponding set of menu preference pairs* $\hat{\mathcal{O}}$ *is rationalizable.*

Recall that the rationalizability of $\hat{\mathcal{O}}$ is characterized by the never-covered property and this property could be checked using Algorithm I (see Section 3.3). We now revisit Example 8 to illustrate how Theorem 7 applies in that case.

Example 8 (Continued)**.** Let $M' = X \backslash z$ and $M'' = X \backslash w$. In this case, $\cup_{D \in \Sigma} f(D) = \{x, y\}$. The set $\bar{\mathcal{O}}^W$ is empty and $\hat{\mathcal{O}} = \hat{\mathcal{O}}^S$ consists of the following menu preference pairs (where the menu on the left is strictly preferred to the one on the right).

$$\begin{aligned} \bar{A}_{y,X} &= \Big\{(y, x), (y, z), (y, w)\Big\}, & \bar{A}_{x,X} &= \Big\{(x, y), (x, z), (x, w)\Big\}; \\ \bar{A}_{x,M'} &= \Big\{(x, y), (x, w)\Big\}, & \bar{A}_{y,M'} &= \Big\{(y, x), (y, w)\Big\}; \\ \bar{A}_{x,M''} &= \Big\{(x, y), (x, z)\Big\}, & \bar{A}_{y,M''} &= \Big\{(y, x), (y, z)\Big\}. \end{aligned}$$

By Corollary 1, $\hat{\mathcal{O}}$ is not rationalizable since

$$\Big(\bar{A}_{y,X} \cup \bar{A}_{x,M'} \cup \bar{A}_{x,M''}\Big) \subseteq \Big(\bar{A}_{x,X} \cup \bar{A}_{y,M'} \cup \bar{A}_{y,M''}\Big).$$

By Theorem 7, (Σ, f) is not rationalizable under the minimax regret model.

[28] Bear in mind we assume that X is finite, so rationalization by a preference coincides with rationalization by a utility function.

Appendix

Proof of Theorem 1. We have argued in the main text that Statement (1) implies Statement (2). In what follows, we show that Statement (2) implies Statement (3) and that Statement (3) implies Statement (1).

Statement (2) $\Rightarrow$ Statement (3): Suppose that $\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$. We shall explicitly provide a way of selecting x^t in A^t for each $t \in T$ such that $\{x^t\}_{t\in T}$ is a no-cycling selection under $\trianglerighteq$.

For ease of notation, let us denote by $\mathcal{E}(T')$ the set of alternatives that are revealed to be dominated through the procedure of iterated exclusion of dominated observations, i.e.,

$$\mathcal{E}(T') := B\,(T')^{\downarrow\downarrow} \bigcup B\,((T' \cap S) \cup \Phi(T'))^{\downarrow}.$$

Since $\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$, for any nonempty $T' \subseteq T$, $\Phi(T')$ is strict subset of T', which implies that $A\,(T') \setminus \mathcal{E}(T') \neq \emptyset$.

Let $T_1 := T$ and $S_1 := A(T_1) \setminus \mathcal{E}(T_1)$. We proceed by induction. Suppose that we have constructed T_k and S_k for some $k \geq 1$. If $T_k \neq \emptyset$, we construct T_{k+1} and S_{k+1} as follows:

$$\begin{aligned} T_{k+1} &:= \Phi(T_k) = \{t \in T_k : A^t \subseteq \mathcal{E}(T_k)\}, \text{ and} \\ S_{k+1} &:= A(T_{k+1}) \setminus \mathcal{E}(T_{k+1}). \end{aligned}$$

Since $\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$, if $T_k \neq \emptyset$, then $T_{k+1} = \Phi(T_k)$ is a strict subset of T_k and $S_k = A(T_k) \setminus \mathcal{E}(T_k) \neq \emptyset$. The construction stops when $T_{k^*} \neq \emptyset$ and $T_{k^*+1} = \emptyset$ for some k^*.

We are now ready to select x^t in A^t for each $t \in T$ such that $\{x^t\}_{t\in T}$ is a no-cycling selection under $\trianglerighteq$. For each $1 \leq k \leq k^*$, let $V_k := T_k \setminus T_{k+1}$ denote the collection of observations that are eliminated when constructing T_{k+1} from T_k. Clearly, $\{V_k\}_{k=1}^{k^*}$ is a partition of T. By definition, for each k and each $t \in V_k = T_k \setminus T_{k+1}$, we have $A^t \setminus \mathcal{E}(T_k) \neq \emptyset$ and hence $A^t \cap S_k = A^t \cap (A(T_k) \setminus \mathcal{E}(T_k)) \neq \emptyset$.

For each k and each $t \in V_k = T_k \setminus T_{k+1}$, we pick an arbitrary $x^t \in A^t \cap S_k$. We proceed to verify that the revealed preference relations defined on $\{x^t\}_{t\in T}$ obey the no-cycling property. Let $k(t)$ be the corresponding index k such that $t \in V_k$. It suffices to show that (1) $x^t R x^{t'}$ implies that $k(t) \leq k(t')$; and (2) $x^t P x^{t'}$ implies that $k(t) < k(t')$. Suppose that $x^t R x^{t'}$ but $k(t) > k(t')$. Then $t \in \Phi(T_{k(t')})$ due to the construction of $\{V_k\}_{k=1}^{k^*}$. It follows that $A^t \subseteq \mathcal{E}(T_{k(t')})$ and

$B^{t\downarrow} \subseteq \mathcal{E}(T_{k(t')})$. Since $x^t R x^{t'}$, we have $x^{t'} \in B^{t\downarrow} \subseteq \mathcal{E}(T_{k(t')})$, which contradicts with $x^{t'} \in S_{k(t')} = A(T_{k(t')}) \setminus \mathcal{E}(T_{k(t')})$. Hence, $x^t R x^{t'}$ implies $k(t) \leq k(t')$. Suppose that $x^t P x^{t'}$ but $k(t) \geq k(t')$. If $k(t) > k(t')$, then we have the same contradiction as argued above. If $k(t) = k(t') = k$ for some k, then both x^t and $x^{t'}$ belong to S_k. Since $S_k = A(T_k) \setminus \mathcal{E}(T_k)$ and $B(T_k)^{\downarrow\downarrow} \cup B(T_k \cap S)^{\downarrow} \subseteq \mathcal{E}(T_k)$, we have

$$x^t, x^{t'} \in S_k \subseteq A(T_k) \setminus \left(B(T_k)^{\downarrow\downarrow} \cup B(T_k \cap S)^{\downarrow}\right).$$

But this is impossible since $x^t P x^{t'}$ implies that either $x^{t'} \in B^{t\downarrow\downarrow}$ or $t \in S$ and $x^{t'} \in B^{t\downarrow}$. Hence, $x^t P x^{t'}$ implies $k(t) < k(t')$.

Statement (3) $\Rightarrow$ Statement (1): Suppose that $\mathcal{M}$ admits a no-cycling selection under $\trianglerighteq$, $\{x^t\}_{t\in T}$. Let R^* be the binary relation on X where $\hat{x}\, R^*\, \hat{y}$ if there is $t \in T$ such that $\hat{x} = x^t$ and $\hat{y} \in B^t$. Let $\succsim^*$ be the transitive closure of the binary relation $R^* \cup \trianglerighteq$. By the Szpilrajn's extension theorem (see Szpilrajn (1930)), $\succsim^*$ admits an extension $\succsim$.[29]

We claim that the preference $\succsim$ has two properties: (1) it rationalizes the data set and (2) it extends $\trianglerighteq$. It follows from the construction that $x^t \succsim B^t$ for all $t \in T$. Thus, to show (1), we only need to show that $x^t \succ B^t$ for $t \in S$. Suppose to the contrary that for some $t' \in S$, $x^{t'} \sim y$ for some $y \in B^{t'}$. Given that $\succsim$ extends $\succsim^*$, this can only occur if $y \succsim^* x^{t'}$. This means there is $t'' \in T$ such that $y \trianglerighteq x^{t''} \succsim^* x^{t'}$. Therefore we obtain $x^{t'}\, P\, x^{t''} \succsim^* x^{t'}$, which is excluded by the no-cycling property. This completes the proof of (1). To show (2), note that $x \succsim y$ if $x \trianglerighteq y$ by construction, so it remains to show that $x \succ y$ if $x \triangleright y$. Suppose instead we have $x \triangleright y$ but $x \sim y$. This can only occur if $y \succsim^* x$. Since $\trianglerighteq$ is a preorder, if $x \triangleright y$ and $y \succsim^* x$, there must be t', $t'' \in T$ such that $y \trianglerighteq x^{t''} \succsim^* x^{t'} \trianglerighteq x$. So we obtain $x^{t'} \trianglerighteq x \triangleright y \trianglerighteq x^{t''}$, which means that $x^{t'}\, P\, x^{t''}$, which is incompatible with $x^{t''} \succsim x^{t'}$, given the no-cycling property. $\square$

Proof of Proposition 1. By Theorem 1, $\mathcal{M}$ is $\trianglerighteq$-rationalizable if and only if it satisfies the never-covered property under $\trianglerighteq$. Thus, it suffices to show that $\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$ if and only if Algorithm I outputs $\trianglerighteq$-*Rationalizable*.

(The only if-part) If $\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$, then $\Phi(T') \neq T'$ for any nonempty $T' \subseteq T$. Thus, Case (c) never occurs when we run Algorithm I on this data set. Furthermore, T^k is strictly decreasing in k in the

[29] This means that there is a complete preorder $\succsim$ such that $x \succsim y$ if $x \succsim^* y$ and $x \succ y$ if $x \succ^* y$ where $\succ$ is the asymmetric part of $\succsim$ and $\succ^*$ the asymmetric part of $\succsim^*$.

set inclusion sense, and $T^{k^*} = \emptyset$ for some k^*. Therefore, Algorithm I outputs $\trianglerighteq$-*Rationalizable*.

(The if-part) Conversely, suppose that Algorithm I outputs $\trianglerighteq$-*Rationalizable*. We then have a sequence of subsets of T, $\{T^0, T^1, \ldots, T^{k^*}\}$, where $T^k = \Phi(T^{k-1}) \subsetneq T^{k-1}$ for all $k = 1, 2, \ldots, k^*$ and $T^{k^*} = \emptyset$. For any nonempty $T' \subseteq T$, there exists some k such that $T' \subseteq T^k$ and $T' \not\subseteq T^{k+1}$. It is straightforward to verify that the operator $\Phi(\cdot)$ is monotonically increasing in the set inclusion sense. As such, $\Phi(T') \subseteq \Phi(T^k) = T^{k+1}$. Since $T' \not\subseteq T^{k+1}$, we have $\Phi(T') \neq T'$. Thus, the data set satisfies the never-covered property under $\trianglerighteq$. $\square$

Proof of Theorem 2. By Theorem 1, Statements (1) and (2) are equivalent, and obviously (3) implies (1). So it remains to show that (2) implies (3).

To the original data set $\mathcal{M}$ we add the observations $\left\{(A^{at}, A^{at})\right\}_{t\in T}$ and $\left\{(B^{bt}, B^{bt})\right\}_{t\in T}$, where $A^{at} = A^t$ and $B^{bt} = B^t$ for each $t \in T$. Consider the augmented data set

$$\mathcal{M}^* = \left\{(A^t, B^t)\right\}_{t\in T} \bigcup \left\{(A^{at}, A^{at})\right\}_{t\in T} \bigcup \left\{(B^{bt}, B^{bt})\right\}_{t\in T},$$

where all the added observations are considered weak preferences. Notice that $\mathcal{M}$ is nicely rationalizable if and only if $\mathcal{M}^*$ is rationalizable.

We denote a typical observation of $\mathcal{M}^*$ by z; the observation z may be in T, in $aT := \{a1, a2, \ldots, at, \ldots\}$, or in $bT := \{b1, b2, \ldots, bt, \ldots\}$. Let $Z := T \cup aT \cup bT$. Suppose that $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t\in T}$ satisfies the never-covered property, i.e., for any $T' \subseteq T$, the set of dominated observations at T', $\Phi(T')$, satisfies $\Phi(T') \neq T'$. We claim that this implies that $\mathcal{M}^*$ also satisfies the never-covered property, which will guarantee (by Theorem 1) that $\mathcal{M}^*$ is rationalizable.

Given Z', a nonempty subset of Z, we denote its dominated observations by $\Phi^*(Z')$. We need to show that $\Phi^*(Z') \neq Z'$. Suppose that $Z' \cap S = \emptyset$, where $S \subseteq T$ is the set of observations with strict preferences. Then $\Phi^*(Z') = \emptyset$ and so obviously $\Phi^*(Z') \neq Z'$. Now suppose that $Z' \cap S$ is nonempty, which means that $Z' \cap T$ is also nonempty. Since $\mathcal{M}$ satisfies the never-covered property, $(Z' \cap T) \setminus \Phi(Z' \cap T)$ is nonempty. It is straightforward to check in this case that if $\hat{t} \in (Z' \cap T) \setminus \Phi(Z' \cap T)$ then $\hat{t} \in Z' \setminus \Phi^*(Z')$. Therefore, $Z' \setminus \Phi^*(Z')$ is nonempty since $(Z' \cap T) \setminus \Phi(Z' \cap T)$ is nonempty. We conclude that $\mathcal{M}^*$ satisfies the never-covered property. $\square$

Proof of Theorem 3. The equivalence of Statements (1) and (2) follows from Theorem 1 and obviously (3) implies (1). It suffices to show that Statement (2) implies (3). By Theorem 1, since $\mathcal{M}$ satisfies the never-covered property under $\trianglerighteq$, it admits a no-cycling selection $\{x^t\}_{t\in T}$ under $\trianglerighteq$. From the proof of Theorem 1, we know that any preference that extends $\text{tran}(R^* \cup \trianglerighteq)$ (the transitive closure of $R^* \cup \trianglerighteq$) will rationalize the data and extend $\trianglerighteq$. It remains to show that there is a preference representable by a continuous utility function that extends $\text{tran}(R^* \cup \trianglerighteq)$. By Levin's Theorem, such an extension exists so long as $\text{tran}(R^* \cup \trianglerighteq)$ is a closed preorder. That is indeed the case (see the proof of this claim contained in the proof of Theorem 2 in Nishimura et al. (2017)) and it follows from the compactness of B^t for all $t \in T$ and the finite number of observations. □

Proof of Theorem 4. Statement (3) obviously implies (1). To show that (1) implies (2), suppose that $\mathcal{M}$ is rationalized by a locally nonsatiated preference $\succsim$ and $x^t \in L(p^t)$ satisfies $x^t \succsim L(q^t)$ for all $t \in T$ and $x^t \succ L(q^t)$ for all $t \in S$. Let $\hat{x} = \max(\{x^t\}_{t\in T'}; \succsim)$. Denoting $L_p(T') = \bigcup_{t\in T'} L(p^t)$ and $L_q(T')$ in an analogous fashion, we note that $L_p(T')$ cannot be covered by $L_q(T')^{\downarrow\downarrow} \bigcup L_q(T' \cap S)^{\downarrow}$ since the former contains $\hat{x}$ and the latter does not. Indeed, $\hat{x} \succ L_q(T' \cap S)^{\downarrow} = L_q(T' \cap S)$, and so $\hat{x}$ is not in $L_q(T' \cap S)$. Nor can $\hat{x}$ be in $L_q(T')^{\downarrow\downarrow} = L_q(T')^o$ (the interior of $L_q(T')$) because $\hat{x} \succsim L_q(T')$ and the preference is locally nonsatiated. Taking it one step further, we know that $\hat{x}$ is not in $L_q(T')^o \bigcup L_q((T' \cap S) \cup \Phi^1(T'))$, where

$$\begin{aligned} \Phi^1(T') &:= \left\{t \in T' : L(p^t) \subseteq L_q(T')^{\downarrow\downarrow} \bigcup L_q(T' \cap S)^{\downarrow}\right\} \\ &= \left\{t \in T' : L(p^t) \subseteq L_q(T')^o \bigcup L_q(T' \cap S)\right\}. \end{aligned}$$

This is because $\hat{x} \succ L_q((T' \cap S) \cup \Phi^1(T'))$. Repeating this argument, we eventually conclude that $\hat{x}$ is in $A(T')$ but not in $L_q(T')^o \bigcup L_q((T' \cap S) \cup \Phi(T'))$ and hence $\Phi(T') \neq T'$.

To show that (2) implies (3), note that Theorem 3 guarantees that there is a strictly increasing and continuous utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ that rationalizes $\mathcal{M}$. So it suffices to show that there is a utility function $\hat{u}$ that also rationalizes $\mathcal{M}$, which has the additional property of concavity. Each budget, $L(p^t)$ is compact and so it has an optimum under u which we denote by x^t (if there are multiple optimal alternatives we may pick any one of them); similarly we denote the optimum bundle in $L(q^t)$ by y^t. Since u is strictly increasing, $p^t \cdot \bar{x}^t = q^t \cdot y^t = 1$, i.e., the optimal bundle is on the budget plane and not just the budget set. Let $\succsim$ denote the preference (i.e, the complete preorder) over $\{x^t\}_{t\in T} \cup \{y^t\}_{t\in T}$ induced by u. Since it is generated by u, the notional data set $\mathcal{N} = \{(x^t, p^t)\}_{t\in T} \cup \{(y^t, q^t)\}_{t\in T}$ is cyclically consistent in the sense of Afriat (1967) (equivalently, obeys the generalized axiom of revealed preference in the sense of Varian (1982)). The preference $\succsim$ is a completion of the revealed preference relations generated by $\mathcal{N}$ and defined on $\{x^t\}_{t\in T} \cup \{y^t\}_{t\in T}$;

by Afriat's Theorem, there is a strictly increasing, continuous and *concave* utility function $\hat{u}$ such that $\hat{u}(x^t) \geq \hat{u}(x)$ for all $x \in L(p^t)$, $\hat{u}(y^t) \geq \hat{u}(x)$ for all $x \in L(q^t)$, and $\hat{u}(x^t) \geq (>)\, u(y^t)$ if $x^t \succsim (\succ)\, y^t$ (where $\succ$ is the asymmetric part of $\succsim$). Thus $\hat{u}$ also rationalizes $\mathcal{M}$. $\square$

Proof of Corollary 2. Obviously Statement (3) implies Statement (1), so we need only show that Statement (1) implies Statement (2) and the latter implies Statement (3).

To see that Statement (1) implies (2), suppose $\mathcal{M}$ is rationalized by a preference $\succsim$. Let $T' \subseteq T$. For each $t \in T'$, there is $x^t \in L(p^t)$ such that $x^t \succ L(q^t)$. If we choose $t^* \in T'$ such that $x^{t^*} \succsim x^t$ for all $t \in T'$, then $x^{t^*} \succ x$ for all $x \in \bigcup_{s \in T'} L(q^s)$. Let $q \in \text{conv}\big(\{q^s\}_{s \in T'}\big)$. For any x such that $q \cdot x \leq 1$, there must exist $q^{\underline{s}}$, with $\underline{s} \in T'$, such that $x \cdot q^{\underline{s}} \leq 1$; in other words $x \in L(q^{\underline{s}})$. Thus $x^{t^*} \succ L(q)$ and so $x^{t^*} \notin L(q)$, which implies that $p^{t^*} \ngeq q$.[30]

We know from Theorem 4 that to show that Statement (2) implies Statement (3), it suffices to show that the never-covered property holds; since $T = S$, this simply requires $\bigcup_{t' \in T'} L(p^{t'}) \nsubseteq \bigcup_{t' \in T'} L(q^{t'})$ for any $T' \subseteq T$. Given such a T', let $P := \Big\{p \in \mathbb{R}^n_+ : p \geq q \text{ for some } q \in \text{conv}\big(\{q^s\}_{s \in T'}\big)\Big\}$. Clearly, P is closed and convex. Statement (2) guarantees that there is $t^* \in T'$ such that $p^{t^*} \notin P$. By the separating hyperplane theorem, there exists a vector $r \in \mathbb{R}^n$ and a number b where $r \neq 0$ such that $p^{t^*} \cdot r = b < p \cdot r$ for any $p \in P$. It is easy to verify that $r \geq 0$. Since $r \neq 0$, we have $b = p^{t^*} \cdot r > 0$. Let $r' = r/b$. We have $p^{t^*} \cdot r' = 1 < p \cdot r'$ for all $p \in P$. In words, r' is affordable at the price vector p^{t^*} but not under any q^s. Therefore, $L(p^{t^*}) \nsubseteq \bigcup_{s \in T'} L(q^s)$. $\square$

Proof of Theorem 5. We skip the proof that Statement (1) implies (2), which is straightforward and similar to the argument given in Section 4 for the claim that (1) implies (2) in Theorem 4. It is also obvious that (3) implies (1). So it remains for us to show that (2) implies (3). An appeal to Theorem 3 guarantees that there is x^t (for each $t \in T$) and a strictly increasing and continuous utility function $\tilde{u} : \mathbb{R}^n_+ \to \mathbb{R}$ such that $\tilde{u}(x^t) \geq \tilde{u}(x)$ for all $x \in L(p^t)$. Therefore the notional data set $\big\{(x^t, p^t)\big\}_{t \in T}$ must satisfy cyclical consistency (equivalently GARP). By Afriat's Theorem, there is a strictly increasing, continuous, and concave utility function u such that $u(x^t) \geq u(x)$ for all $x \in L(p^t)$ for all $t \in T$. $\square$

Proof of Theorem 6. We have already shown that Statements (1) and (2) are equivalent, and that Statement (1) implies Statement (3). It remains to show

[30] Notice that unlike Theorem 4, we do not require the preference to be locally non-satiated in Statement (1), because all the menu preferences in the current setting are strict.

that Statement (3) implies Statement (2). Suppose, contrary to Statement (2), that there is $B \in \Sigma$ and $x \in f(B)$, such that $\mathcal{M}_{B,x}$ is not rationalizable by a strict preference. Abusing notation somewhat, we shall identify the observations in $\mathcal{M}_{B,x}$ with the elements of Σ. By Corollary 1, the never-covered property (2) is violated for some subset Σ' of the observations in $\mathcal{M}_{B,x}$. If $B \notin \Sigma'$, then we have $\cup_{A\in\Sigma'} f(A) \subseteq \cup_{A\in\Sigma'} g(A)$ but this is impossible because it is excluded by (6) (see footnote 25). So we consider the case where $B \in \Sigma'$ and let $\Sigma'' = \Sigma' \setminus B$. Since (2) is violated,

$$\{x\} \cup \Big(\cup_{A\in\Sigma''} \ f(A)\Big) \subseteq (B \setminus x) \cup \Big(\cup_{A\in\Sigma''} \ g(A)\Big). \tag{14}$$

However, by (6) again, we have $\cup_{A\in\Sigma''} f(A) \not\subseteq \cup_{A\in\Sigma''} g(A)$. This implies that

$$\Big(\cup_{A\in\Sigma''} \ f(A) \setminus \cup_{A\in\Sigma''} g(A)\Big) \subseteq B$$

and so $f(B) \cap (\cup_{A\in\Sigma''} g(A)) = \emptyset$ (by (6)). In partcular, $x \notin \cup_{A\in\Sigma''} g(A)$, which means that (14) is impossible. And so we obtain a contradiction. □

Proof of Proposition 2. We first make three easy-to-check observations. (i) $N(\Sigma', B) = \emptyset$ if and only if $(\cup_{A\in\Sigma'} f(A)) \setminus (\cup_{A\in\Sigma'} g(A)) \subseteq B$. (ii) If $N(\Sigma', B) = \emptyset$ and $N(\Sigma'', B) = \emptyset$, then $N(\Sigma' \cup \Sigma'', B) = \emptyset$; this guarantees that there is Σ^* (possibly empty) such that $N(\Sigma^*, B) = \emptyset$ and if $N(\Sigma', B) = \emptyset$ then $\Sigma' \subseteq \Sigma^*$. It follows that condition (6) need only be checked for Σ^*. (iii) Suppose that for some Σ', we find that $N(\Sigma', B)$ is nonempty and, for some $\bar{A} \in \Sigma'$, $f(\bar{A}) \cap N(\Sigma', B)$ is nonempty. Then $\bar{A} \notin \Sigma''$ if $\Sigma'' \subseteq \Sigma'$ and $N(\Sigma'', B) = \emptyset$.

It follows from these observations, that if $N(\Sigma, B)$ is nonempty (so $\Sigma^* \neq \Sigma$), then $\Sigma^1 = \{A \in \Sigma : f(A) \cap N(\Sigma, B) = \emptyset\}$ is a *strict* subset of Σ that (from observation (iii) above) contains Σ^*. If $N(\Sigma^1, B)$ is nonempty, then the algorithm calculates (in step (b)) $\Sigma^2 = \{A \in \Sigma^1 : f(A) \cap N(\Sigma^1, B) = \emptyset\}$. Again, Σ^2 is a strict subset of Σ^1 and contains Σ^*. Eventually, for some m, we have $N(\Sigma^m, B) = \emptyset$, at which point we conclude that $\Sigma^m = \Sigma^*$. The algorithm then requires us to check if (6) holds. □

Proof of Corollary 3. The only if-part is trivial. We prove the if-part below, by showing that condition (6) in Theorem 6 follows from the Chernoff and Aizerman axioms.

Claim 1. *For any $A \in \mathcal{X}$, if $x \in g(A)$, then $f(f(A) \cup x) = f(A)$.*

Proof. Since $\big(f(A) \cup x\big) \subseteq A$, the Chernoff axiom implies that $f(A) \subseteq f(f(A) \cup x)$. Since $f(A) \subseteq \big(f(A) \cup x\big) \subseteq A$, the Aizerman axiom implies that $f(f(A) \cup x) \subseteq f(A)$. Thus, $f(f(A) \cup x) = f(A)$.

Claim 2. *For any $A, B \in \mathcal{X}$, if $f(A) \subseteq B$, then $f(B) \cap g(A) = \emptyset$.*

Proof. Suppose that there exists some $x \in f(B) \cap g(A)$. Since $x \in g(A)$, by Claim 1, x is not chosen in $f(A) \cup x$. The Chernoff axiom implies that x is not chosen in any superset of $f(A) \cup x$. In particular, $x \notin f(B)$. We have a contradiction.

Claim 3. *Condition (6) holds, i.e., for any nonempty $\mathcal{X}' \subseteq \mathcal{X}$ and $B \in \mathcal{X}$, if $\Big(\cup_{A\in\mathcal{X}'} f(A) \setminus \cup_{A\in\mathcal{X}'} g(A)\Big) \subseteq B$, then $f(B) \cap \Big(\cup_{A\in\mathcal{X}'} g(A)\Big) = \emptyset$.*

Proof. Fix an arbitrary nonempty $\mathcal{X}' \subseteq \mathcal{X}$. By the Chernoff axiom, $g(C) \subseteq g\Big(\cup_{A\in\mathcal{X}'} A\Big)$ for all $C \in \mathcal{X}'$. Therefore, $\cup_{A\in\mathcal{X}'} g(A) \subseteq g\Big(\cup_{A\in\mathcal{X}'} A\Big)$. We then have

$$\begin{aligned} f\Big(\cup_{A\in\mathcal{X}'} A\Big) &= \cup_{A\in\mathcal{X}'} A \setminus g\Big(\cup_{A\in\mathcal{X}'} A\Big) \\ &\subseteq \cup_{A\in\mathcal{X}'} A \setminus \cup_{A\in\mathcal{X}'} g(A) \\ &= \cup_{A\in\mathcal{X}'} f(A) \setminus \cup_{A\in\mathcal{X}'} g(A). \end{aligned}$$

Thus, if $\Big(\cup_{A\in\mathcal{X}'} f(A) \setminus \cup_{A\in\mathcal{X}'} g(A)\Big) \subseteq B$ for some $B \in \mathcal{X}$, we must have $f\Big(\cup_{A\in\mathcal{X}'} A\Big) \subseteq B$. By Claim 2, $f(B) \cap g\Big(\cup_{A\in\mathcal{X}'} A\Big) = \emptyset$. Since $\cup_{A\in\mathcal{X}'} g(A) \subseteq g\Big(\cup_{A\in\mathcal{X}'} A\Big)$, we obtain $f(B) \cap \Big(\cup_{A\in\mathcal{X}'} g(A)\Big) = \emptyset$.

By Theorem 6, $(\mathcal{X}, f)$ is rationalizable by multiple preferences. $\square$

Proof of Theorem 7. We have already shown that if (Σ, f) is rationalizable by the minimax regret model then $\hat{\mathcal{O}}$ is rationalizable. Turning to the converse, we first observe that if $\hat{\mathcal{O}}$ is rationalizable then its superset $\bar{\mathcal{O}}$ is also rationalizable. Indeed, let T' be a subset of observations in $\bar{\mathcal{O}}$ and suppose it contains an observation $(\bar{A}_{\tilde{x},M}, \bar{A}_{x^*,M})$ where $\tilde{x} \notin \cup_{D\in\Sigma} f(D)$. Notice that this observation cannot be in $\Phi(T')$, the set of revealed dominated observations, which implies that $T' \setminus \Phi(T')$ is nonempty. So in checking whether the never-covered property holds for $\bar{\mathcal{O}}$, we need only consider those subsets of observations T' which do not contain observations of that type. Thus, $\bar{\mathcal{O}}$ satisfies the never-covered property if $\hat{\mathcal{O}}$ satisfies the never-covered property; equivalently, $\bar{\mathcal{O}}$ is rationalizable if $\hat{\mathcal{O}}$ is rationalizable.

Now suppose $\bar{\mathcal{O}}$ is rationalized by the preference $\succsim$ on $\bar{X}$. It remains for us to show that, for any preference relation $\succsim$ defined over $\bar{X}$, there exists a finite set U of utility functions such that for any $(x, y), (z, w) \in \bar{X}$, $\phi_U(x, y) \geq \phi_U(z, w)$ if and only if $(x, y) \succsim (z, w)$. Since $\bar{X}$ is finite, we can construct a function $\beta : \bar{X} \to (1, 2)$ such that $\beta(x, y) \geq \beta(z, w)$ if and only if $(x, y) \succsim (z, w)$. We now construct a finite set of utility functions $U = \{u_{x,y}\}_{(x,y)\in\bar{X}}$, where the utility functions are indexed by

$(x, y) \in \bar{X}$. For each $(x, y) \in \bar{X}$, let

$$u_{x,y}(z) = \begin{cases} 0, & \text{if } z = x; \\ \beta(x, y), & \text{if } z = y; \\ \frac{\beta(x,y)}{2}, & \text{otherwise.} \end{cases}$$

Obviously, $u_{x,y}(y) - u_{x,y}(x) = \beta(x, y) > 1$. We claim that $u_{z,w}(y) - u_{z,w}(x) < 1$ if $(z, w) \neq (x, y)$. First consider the case in which $w \neq y$. It follows from the construction of the utility functions and the β function that

$$u_{z,w}(y) - u_{z,w}(x) \leq u_{z,w}(y) \leq \frac{\beta(z, w)}{2} < 1.$$

Next, we consider the case in which $w = y$. Since $(z, w) \neq (x, y)$, we have $z \neq x$. By the construction of the utility functions,

$$u_{z,w}(y) - u_{z,w}(x) = \beta(z, w) - \frac{\beta(z, w)}{2} = \frac{\beta(z, w)}{2} < 1.$$

Therefore, we can conclude that

$$\phi_U(x, y) = \max_{u \in U} \left\{ u(y) - u(x) \right\} = u_{x,y}(y) - u_{x,y}(x) = \beta(x, y).$$

We have constructed a finite set U of utility functions such that for any $(x, y), (z, w) \in \bar{X}$,

$$\phi_U(x, y) \geq \phi_U(z, w) \iff \beta(x, y) \geq \beta(z, w) \iff (x, y) \succsim (z, w).$$

(Σ, f) can thus be rationalized by the minimax regret model. $\square$

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Online Appendix
to
"A Theory of Revealed Indirect Preference"

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This Online Appendix is divided into three sections. Section A discusses further results on the never-covered property and extends Corollary 2 in the main paper to the case where menu preferences are allowed to be weak. Section B discusses the connection between our work and that of de Clippel and Rozen (2021) on upper contour rationalization. Section C contains an empirical analysis of the experimental data set from Choi et al. (2007). We use our algorithm for checking the never-covered property to calculate the perturbation index for each subject in that experiment.

A More results on the never-covered property

By definition, a data set of menu preference pairs $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t \in T}$ satisfies the never-covered property under $\trianglerighteq$ if, for any nonempty $T' \subseteq T$, $\Phi(T')$ is a strict subset of T'. Obviously, this means that there exists some $\Psi \subsetneq T'$ such that for any $t \in T' \setminus \Psi$, $A^t \not\subseteq B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Psi)^{\downarrow}$. To see this, we could simply set $\Psi = \Phi(T')$. In fact, as the next proposition states formally, we could characterize $\Phi^(T')$ as the smallest set of observations in T' with this property.

Proposition A1. *Consider the data set* $\mathcal{M} = \left\{(A^t, B^t)\right\}_{t \in T}$.

(1) For any nonempty $T' \subseteq T$, *if there exists* $\Psi \subsetneq T'$ *such that for any* $t \in T' \setminus \Psi$,
$$A^t \not\subseteq B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Psi)^{\downarrow},$$
then $\Phi(T') \subseteq \Psi$.

(2) If for any nonempty $T' \subseteq T$, *there exists* $\Psi \subsetneq T'$ *such that for any* $t \in T' \setminus \Psi$,
$$A^t \not\subseteq B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Psi)^{\downarrow},$$
then $\mathcal{M}$ *satisfies the never-covered property under* $\trianglerighteq$.

Proof. We prove the first statement below. The second statement is an immediate implication of the first statement. Fix a nonempty $T' \subseteq T$ and $\Psi \subsetneq T'$ such that for any $t \in T' \setminus \Psi$,
$$A^t \not\subseteq B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Psi)^{\downarrow}.$$
To show that $\Phi(T') \subseteq \Psi$, we proceed by induction. Obviously,
$$\Phi^0(T') = \emptyset \subseteq \Psi,$$

$$\Phi^1(T') = \left\{t \in T' : A^t \subseteq B(T')^{\downarrow\downarrow} \bigcup B(T' \cap S)^{\downarrow}\right\} \subseteq \Psi.$$

Suppose that we have $\Phi^m(T') \subseteq \Psi$ for some m. It follows that

$$B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Phi^m(T'))^{\downarrow} \subseteq B(T')^{\downarrow\downarrow} \bigcup B((T' \cap S) \cup \Psi)^{\downarrow}.$$

Thus, $t \in T' \setminus \Psi$ implies $t \in T' \setminus \Phi^{m+1}(T')$, which further implies that $\Phi^{m+1}(T') \subseteq \Psi$. By induction, we obtain $\Phi(T') \subseteq \Psi$. □

In Corollary 2 of the main paper, we considered the case where a data set consists of strict preferences between linear budget sets, i.e., $\mathcal{M} = \left\{(L(p^t), L(q^t))\right\}_{t\in T}$ where $T = S$. In this case, we showed that the never-covered property, which guarantees the rationalizability of $\mathcal{M}$ by a preference on $\mathbb{R}^n_+$, can be equivalently stated as the following property which is closer to the quasi-convex property of the indirect utility function: for any nonempty $T' \subseteq T$, there is $t^* \in T'$ such that $p^{t^*} \ngeq q$ for any $q \in \text{conv}\left(\{q^s\}_{s\in T'}\right)$.

The next result applies Proposition A1 to extend Corollary 2 in the main paper to the case in which weak preference between prices are observed, i.e., where W can be nonempty.

Proposition A2. *Consider a set of preferences over budget sets $\mathcal{M} = \left\{(L(p^t), L(q^t))\right\}_{t\in T}$. The data set $\mathcal{M}$ is rationalizable by a strictly increasing, continuous, and concave utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ if and only if the following property holds: for any nonempty $T' \subseteq T$,*

> *there exists $\Phi \subsetneq T'$ and $\epsilon \gg 0$ such that $p \ngeq q$ whenever $p \ll p^t$ for some $t \in T' \setminus \Phi$ and $q \in \text{conv}\left(\{q^s\}_{s\in T'} \bigcup \{q^r - \epsilon\}_{r\in\Phi\cup(T'\cap S)}\right)$.*[1]

Proof. By Theorem 4 of the main paper, it suffices to show that $\mathcal{M}$ satisfies the never-covered property under $\geq$ if and only if the property in Proposition A2 holds. The never-covered property under $\geq$ requires that for any nonempty $T' \subseteq T$, $T' \setminus \Phi(T') \neq \emptyset$. By the definition of $\Phi(T')$, if $t \in T' \setminus \Phi(T')$, then

$$L(p^t) \nsubseteq \left(\bigcup_{t'\in T'} L(q^{t'})^o\right) \bigcup \left(\bigcup_{t'\in(T'\cap S)\cup\Phi(T')} L(q^{t'})\right).$$

We first prove the following lemma.

Lemma 1. *$L(p) \nsubseteq \left(\bigcup_{s=1}^n L(q^s)\right) \cup \left(\bigcup_{r=1}^m L(q^r)^o\right)$ if and only if there exists some $\epsilon \gg 0$ such that for any $\hat{p} \ll p$, $L(\hat{p}) \nsubseteq \left(\bigcup_{s=1}^n L(q^s - \epsilon)\right) \cup \left(\bigcup_{r=1}^m L(q^r)\right)$.*

[1] When $T = S$, the characterizing condition in this corollary reduces to that in Corollary 2 in the main paper. To see this, note that if $T = S$, then the condition in Proposition A2 is equivalent to saying that for any nonempty $T' \subseteq T$, there exists $t \in T'$ and $\epsilon \gg 0$ such that for any $p \ll p^t$ and $q \in \text{conv}\left(\{q^s\}_{s\in T'} \bigcup \{q^r - \epsilon\}_{r\in T'}\right)$, $p \ngeq q$. By continuity, this is equivalent to the condition in Corollary 2.

Proof. (The only if-part) Since $L(p) \nsubseteq \left(\bigcup_{s=1}^{n} L(q^s)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)^o\right)$, there exists some $x \in L(p)$ such that $x \notin \left(\bigcup_{s=1}^{n} L(q^s)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)^o\right)$. Fix such an x. Since $x \in L(p)$, for any $\hat{p} \ll p$, there exists some $\delta \gg 0$ such that $x + \delta \in L(\hat{p})$. Fix such a δ. Since $x \notin \bigcup_{s=1}^{n} L(q^s)$, there exists some sufficiently small $\epsilon \gg 0$ such that $x \notin \bigcup_{s=1}^{n} L(q^s - \epsilon)$, which further implies that $x + \delta \notin \bigcup_{s=1}^{n} L(q^s - \epsilon)$. Since $x \notin \bigcup_{r=1}^{m} L(q^r)^o$, $x + \delta \notin \bigcup_{r=1}^{m} L(q^r)$. Since $x + \delta$ is contained in $L(\hat{p})$ but not in $\left(\bigcup_{s=1}^{n} L(q^s - \epsilon)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)\right)$, we have $L(\hat{p}) \nsubseteq \left(\bigcup_{s=1}^{n} L(q^s - \epsilon)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)\right)$.

(The if-part) Fix $\epsilon \gg 0$ such that for any $\hat{p} \ll p$, $L(\hat{p}) \nsubseteq \left(\bigcup_{s=1}^{n} L(q^s - \epsilon)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)\right)$. Clearly, for each $\theta \in (\frac{1}{2}, 1)$, $\theta p \ll p$, and there exists $x^\theta \in L(\theta p)$ such that $x^\theta \notin \left(\bigcup_{s=1}^{n} L(q^s - \epsilon)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)\right)$. Note that $\{x^\theta\}_{\theta \in (\frac{1}{2},1)} \subseteq L(\frac{1}{2}p)$, which is compact. Therefore, $\{x^\theta\}_{\theta \in (\frac{1}{2},1)}$ has a convergent sequence. Let x denote the limit of this sequence. Since $x^\theta \in L(\theta p)$, $x \in L(p)$. Furthermore, (1) $x \notin \left(\bigcup_{s=1}^{n} L(q^s)\right)$, otherwise $x \in L(q^s - \epsilon)^o$ for some s, which implies that $x^\theta \in L(q^s - \epsilon)^o$ for some θ and some s; and (2) $x \notin \left(\bigcup_{r=1}^{m} L(q^r)^o\right)$, otherwise $x^\theta \in L(q^r)$ for some r and some θ. Since x is contained in $L(p)$ but not in $\left(\bigcup_{s=1}^{n} L(q^s)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)^o\right)$, we have $L(p) \nsubseteq \left(\bigcup_{s=1}^{n} L(q^s)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)^o\right)$. □

It is easy to see that $L(\hat{p}) \nsubseteq \left(\bigcup_{s=1}^{n} L(q^s - \epsilon)\right) \cup \left(\bigcup_{r=1}^{m} L(q^r)\right)$ if and only if for all $q \in conv\left(\{q^s - \epsilon\}_{s=1}^{n} \cup \{q^r\}_{r=1}^{m}\right)$, $\hat{p} \ngeq q$ (by a separating hyperplane argument similar to the one used in the proof of Corollary 2 of the main paper). The desired result then follows from this and Proposition A1. □

B Upper contour rationalization

As we explain in Section 3.5 of our main paper there is a connection between menu preference rationalization and the upper contour rationalization studied in de Clippel and Rozen (2021) (see de Clippel and Rozen (2012) for an early version). In particular, when X is finite (so that all the relevant subsets in both problems are also finite), the two problems could be thought of as equivalent in the sense that it is always possible to convert one problem into the other.

Indeed, notice that menu A is strictly preferred to menu B if and only if there is an upper contour rationalization of the following set of observations:

$$\left(\{\{y\}\}_{y \in A}, x\right) \text{ for each } x \in B.$$

For example, $\{x, y\}$ is strictly preferred to $\{z, w\}$ if and only if there is an upper contour rationalization of the observations $(\{\{x\}, \{y\}\}, z)$ and $(\{\{x\}, \{y\}\}, w)$. Conversely, suppose we wish to guarantee that there is a set in the collection

$\{A_j\}_{j\in J}$ that is contained in the upper contour set of x; this is equivalent to a rationalization of the following collection of menu preference pairs:

$$\Big(\cup_{j\in J}\{y_j\}, x\Big) \text{ for each } \cup_{j\in J}\{y_j\} \text{ where } y_i \in A_j \text{ for all } j \in J.$$

For example, either $\{x, y\}$ or $\{z\}$ is in the upper contour set of w if and only if there is a rationalization of the following menu preferences: $\{x, z\}$ is preferred to $\{w\}$ and $\{y, z\}$ is preferred to $\{w\}$.

It follows from these observations that it is always possible to convert an upper contour rationalization problem into a menu preference rationalization problem and vice versa when X is finite (but not when it is infinite) and any algorithm developed for one problem could, in principle, be used to solve the other. However, it should also be clear from the conversion procedure we outlined above that there is no general reason for solving either problem in this roundabout fashion, since the converted data set would have more (and in some cases *many* more) observations than the original data set. The two algorithms are best understood as distinct and serving different purposes.

C Never-Covered Property Algorithm

Our objective in this section is to provide an implementation of Algorithm I for checking the never-covered property, which we formulated in Section 3.3 of the main paper. Specifically, we use the algorithm to calculate a measure of the severity of departures from rationality, first outlined in Section 5.3 (Example 6) of the main paper.

C.1 Perturbation Index

Our starting point is a data set $\mathcal{D} = \{(x^t, L(p^t))\}$ where $x^t \in \mathbb{R}^n_+$ is the observed choice from the budget set $L(p^t) = \{x \in \mathbb{R}^n_+ : p^t \cdot x \le 1\}$ (where $p^t \in \mathbb{R}^n_{++}$ are the prevailing prices at observation t and income is normalized at 1 across all observations). Such a data set is (exactly) $\trianglerighteq$-rationalizable if there is a utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ that extends $\trianglerighteq$ such that $u(x^t) \ge u(x)$ for all $x \in L(p^t)$. Afriat's Theorem and its generalizations (see, in particular, Nishimura et al. (2017)), provide necessary and sufficient conditions under which $\mathcal{D}$ is $\trianglerighteq$-rationalizable.

In most empirical settings (with observational or experimental data) it is common for subjects to fail this exact test of rationality and various ways have been proposed to measure the severity of a subject's departure from rationality. Perhaps the most common measure is the *critical cost efficiency index* due to Afriat (1973), which measures the extent to which the revealed preference implications of the data have to be attenuated in order to guarantee rationalizability. We say that $\mathcal{D}$ is $\trianglerighteq$-*rationalizable at efficiency level* $e \in (0,1]$ if there is a utility function u extending $\trianglerighteq$ such that $u(x^t) \geq u(x)$ for all $x \in \mathbb{R}^n_+$ that satisfy $p^t \cdot x \leq e$. Obviously, if $\mathcal{D}$ is $\trianglerighteq$-rationalizable, then it is $\trianglerighteq$-rationalizable at efficiency level 1. The critical cost efficiency index is

$$e^* := \sup\{e : \mathcal{D} \text{ is } \trianglerighteq\text{-rationalizable at efficiency level } e\}.$$

In Section 5.3 (Example 6) of the main paper, we explain how our results could be used to calculate another measure of the departure from rationality by measuring how significantly we need to modify the recorded demand x^t before the data set becomes exactly rationalizable. To be precise, the researcher in our example observes that the consumer chooses the bundle x^t from the budget set $L(p^t)$. To allow for the possibility that x^t was observed with error, the researcher could allow for the true consumption bundle to be in the set

$$A^{t,\kappa} = \{x \in L(p^t) : \ p^t \cdot x = 1 \text{ and } |p^t_i x_i - p^t_i x^t_i| \leq \kappa \text{ for all } i\},$$

where $\kappa \in [0,1]$. In other words, the true expenditure on good i is allowed to deviate from $p^t_i x^t_i$ but not by more than the fraction κ. In experimental settings, where there is no question that x^t is indeed the observed choice, we could interpret κ as a measure of the extent to which we allow the subject to make mistakes in her choice. Whatever the interpretation, we can test whether there is a utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ that extends a given preorder $\trianglerighteq$ and rationalizes the coarse data set

$$\mathcal{O}^\kappa = \left\{\left(A^{t,\kappa}, L(p^t)\right)\right\}_{t\in T}$$

in the sense that

$$\arg\max\{u(x) : x \in L(p^t)\} \cap A^{t,\kappa} \neq \emptyset \ \text{ for all } t \in T.$$

In other words, if $\mathcal{O}^\kappa$ is rationalizable, we know that there are bundles $\tilde{x}^t$, such that $|p^t_i \tilde{x}^t_i - p^t_i x^t_i| \leq \kappa$ for all i and t, and $\widetilde{\mathcal{D}} = \{(\tilde{x}^t, L(p^t)\}_{t\in T}$ is exactly rationalizable. Provided we can determine the rationalizability of $\mathcal{O}^\kappa$, we can calculate the *perturbation index*

$$\kappa^* := \inf\left\{\kappa : \mathcal{O}^\kappa \text{ is } \trianglerighteq\text{-rationalizable}\right\}$$

by binary interpolation. The index gives the smallest perturbation of the observations needed to guarantee that the coarsened data set $\mathcal{O}^\kappa$ is $\trianglerighteq$-rationalizable. Obviously, if $\mathcal{D}$ is exactly rationalizable to begin with, then its perturbation index equals zero.[2]

Relationship with Varian (1985). The idea of relaxing a revealed preference

[2]Notice that if $\kappa = 1$, then the set $A^t = L(p^t)$ and thus $\mathcal{O}^1$ is always rationalizable. Thus the perturbation index is always well-defined.

test to allow for measurement error was also explored by Varian (1985). That paper uses this idea to test the hypothesis that a firm is cost minimizing. Using our language, the introduction of measurement error leads to a coarsening of the data set, where the true choice made by the firm is allowed to be in a ball around the observed choice. Because the hypothesis being tested is different (cost-minimization rather than utility-maximization), the test used in that paper is not related to the never-covered property. To be precise, Varian (1985) assumes that the observer has information on factor prices, factor demand (imperfectly observed) and the output level. In the context of consumption data, the analog to the output level would be the utility level, but that information (even in an ordinal form) is not part of the observations. It is the absence of this information that makes testing the utility-maximization hypothesis (with or without coarsening) a different exercise from testing cost-minimization.

Computing the perturbation index. According to our main result (Theorem 1 in Section 3.2) the rationalizability of $\mathcal{O}^\kappa$ by a utility function extending $\trianglerighteq$ is equivalent to the never-covered property under $\trianglerighteq$. The latter could be checked using Algorithm I set out in Section 3.3. In the empirical application discussed in the next subsection, we focus on two preorders. The first case is where $\trianglerighteq = \geq$, the product order, which is equivalent to requiring the utility function u to be strictly increasing. The second case is $\trianglerighteq = \geq_{sym}$, where $x \geq_{sym} x'$ whenever there is a permutation of the entries of x, which we denote by x^σ, such that $x^\sigma \geq x$. It is straightforward to check that a utility function extends the preorder $\geq_{sym}$ if and only if it is strictly increasing and symmetric. Since $\geq \subset \geq_{sym}$, if $\mathcal{O}^\kappa$ is $\geq_{sym}$-rationalizable then it must be $\geq$-rationalizable and so the perturbation index in the former (more restrictive) model must be weakly greater than in the latter.

Checking whether $\mathcal{O}^\kappa$ is $\trianglerighteq$-rationalizable using Algorithm I requires us to check whether, for a certain set $T' \subseteq T$, we have $T' = \Phi(T')$, where $\Phi(T')$ is the set of revealed dominated observations. The calculation of $\Phi(T')$ in turn uses an iterative procedure set out in Section 3.1, and involves the calculation of increasing subsets of observations $\Phi^1(T')$, $\Phi^2(T')$ and so forth. Implementing that iterative procedure is straightforward because, in our context, checking if an observation belongs to $\Phi^k(T')$ (for a given k) reduces to checking if there is a solution to a system of linear inequalities.

For example, if $\trianglerighteq = \geq$, then $L(p^t)^\downarrow = L(p^t)$ and $L(p^t)^{\downarrow\downarrow} = L(p^t)^o$ (i.e., the interior of $L(p^t)$), so

$$\Phi^1(T') = \left\{ t \in T' : A^t \subseteq \bigcup\nolimits_{t \in T'} L(p^t)^o \right\}.$$

We calculate $\Phi^1(T')$ by checking whether each A^s (for $s \in T'$) belongs to $\bigcup_{t\in T'} L(p^t)^o$ and this in turn can be verified by checking if there is a solution to the system of linear inequalities below, with unknown $z \in \mathbb{R}^n$:

$$\begin{aligned} z &\geq 0 \\ p^s \cdot z &= p^s \cdot x^s \\ |p_i^s z_i - p_i^s x_i^s| &\leq \kappa \quad \text{for each good } i \\ p^t \cdot z &\geq 1, \quad \text{for all } t \in T'. \end{aligned}$$

Clearly, $A^s \subseteq \bigcup_{t\in T'} L(p^t)^o$ if and only if there is *no* solution to this system of linear inequalities. Having calculated $\Phi^1(T')$ we can then calculate $\Phi^2(T')$ (which contains $\Phi^1(T')$), where

$$\Phi^2(T') := \left\{ t \in T' : A^t \subseteq \left(\bigcup_{t\in T'} L(p^t)^o\right) \bigcup \left(\bigcup_{t\in \Phi^1(T')} L(p^t)\right) \right\},$$

$\Phi^3(T')$, and so forth until $\Phi^m(T') = \Phi^{m+1}(T')$ at which point we can check whether $\Phi(T') := \Phi^m(T')$ is a strict subset of T' (as required by the algorithm).

In the case where $\trianglerighteq = \geq_{sym}$, the calculation of $\Phi(T')$ is a bit more elaborate. We confine ourselves to outlining the case where $n = 2$ since that is the case in the application we consider in the next subsection. For this preorder, $L(p^t)^{\downarrow} = L(p^t) \cup L(\tilde{p}^t)$, where $\tilde{p}^t = (b, a)$ if $p^t = (a, b)$ and $L(p^t)^{\downarrow\downarrow} = L(p^t)^o \cup L(\tilde{p}^t)^o$. Then

$$\Phi^1(T') = \left\{ t \in T' : A^t \subseteq \left(\bigcup_{t\in T'} L(p^t)^o\right) \bigcup \left(\bigcup_{t\in T'} L(\tilde{p}^t)^o\right) \right\}.$$

To check if a given observation s is in $\Phi^1(T')$, we solve the following linear system for z:

$$\begin{aligned} z &\geq 0 \\ p^s \cdot z &= p^s \cdot x^s \\ |p_i^s z_i - p_i^s x_i^s| &\leq \kappa \quad \text{for each good } i \\ p^t \cdot z &\geq 1 \quad \text{for all } t \in T \text{ and} \\ \tilde{p}^t \cdot z &\geq 1 \quad \text{for all } t \in T. \end{aligned}$$

We conclude that $s \in \Phi^1(T')$ if there is no solution to this linear system. Having ascertained $\Phi^1(T')$ we proceed in a similar fashion to determine $\Phi^2(T')$ and so on, until we obtain $\Phi(T')$ and could then check if it is a strict subset of T'.

C.2 Empirical Analysis

We study the data collected from the portfolio choice experiment in Choi et al. (2007). The experiment was performed on 93 undergraduate subjects at the University of California, Berkeley. Every subject was asked to make consumption choices on 50 decision problems. In each problem, the subject divided her budget between two

Arrow-Debreu securities, with each security paying 1 token (equivalent to US\$0.50) if the corresponding state was realized, and 0 otherwise. We focus on the symmetric treatment where each state of the world occurred with a commonly known probability of 1/2. This treatment was applied to 47 subjects (subjects ID 201-219 and 301-328). The prices of the Arrow-Debreu were chosen at random (over some compact interval) and varied across problems and subjects, with income normalized at 1 throughout.

For each state $s \in \{1, 2\}$, let x_s denote the demand for the security that pays off in that state and let p_s denote its price. For each subject and in each decision problem $t \in T = \{1, \ldots, 50\}$, the state prices $p^t = (p_1^t, p_2^t)$ were randomly chosen and the subject faced a budget set

$$L(p^t) = \left\{ x \in \mathbb{R}_+^2 : p_1^t x_1 + p_2^t x_2 \leq 1 \right\}.$$

The data set for a subject can be written as $\mathcal{D} = \{(x^t, L(p^t))\}_{t=1}^{50}$, where x^t is the subject's choice in $L(p^t)$.

In calculating the perturbation index, we apply Algorithm I repeatedly across different values of κ, at each stage checking the $\trianglerighteq$-rationalizability of $\mathcal{O}^\kappa$ and obtaining the perturbation index by binary interpolation. We calculate the perturbation indices in the cases where $\trianglerighteq = \geq$ and $\geq_{sym}$ and denote the corresponding indices by κ^* and κ^{**}. Note that since the two states are equiprobable, $\geq_{sym} = \geq_{FSD}$ (see the definition of the latter in Section 3.7 (Example 4) of the main paper); in other words, a subject's utility function respects first order stochastic dominance if and only if it extends $\geq_{sym}$.

As an illustration of how the perturbation index is computed, Table 1 shows the steps involved when Algorithm I is applied to data from Subject 201. The algorithm involves calculating the set of revealed dominated observations $T^1 := \Phi(T)$ and then checking whether $T^1 = T$; if not, it calculates $T^2 := \Phi(T^1)$ and checks whether $T^2 = T^1$; and so forth until either $T^k = T^{k-1}$ (in which case $\mathcal{O}^\kappa$ is not $\trianglerighteq$-rationalizable) or $T^k = \emptyset$ (in which case $\mathcal{O}^\kappa$ is $\trianglerighteq$-rationalizable). T^1, T^2, $T^3 \ldots$ form a nested sequence of sets; the number of elements in each set is indicated in Table 1. We see that when $\kappa = 0.2$, $\mathcal{O}^\kappa$ is not $\geq$-rationalizable because $T^{10} = T^{11}$ and is nonempty while $\mathcal{O}^\kappa$ is $\geq$-rationalizable when $\kappa = 0.3$, because T^{16} is empty. This suggests that κ^* lies between 0.2 and 0.3. Indeed, by binary interpolation, we find that $\kappa^* = 0.2151$.

Similarly, to calculate κ^{**}, we need to check if $\mathcal{O}^\kappa$ is $\geq_{sym}$-rationalizable for different values of κ. Applying our Algorithm I again, we find that $\mathcal{O}^\kappa$ is not $\geq_{sym}$-rationalizable when $\kappa = 0.3$ because $T^9 = T^{10}$ and is nonempty; on the other hand, when $\kappa = 0.4$, $\mathcal{O}^\kappa$ becomes $\geq_{sym}$-rationalizable. Thus κ^{**} is between 0.3 and

	$\trianglerighteq=\geq$		$\trianglerighteq=\geq_{sym}$	
	$\kappa=0.2$	$\kappa=0.3$	$\kappa=0.3$	$\kappa=0.4$
$\lvert T^1\rvert$	48	47	48	48
$\lvert T^2\rvert$	47	46	46	46
$\lvert T^3\rvert$	45	44	45	45
$\lvert T^4\rvert$	41	39	42	41
$\lvert T^5\rvert$	39	36	41	36
$\lvert T^6\rvert$	37	31	39	34
$\lvert T^7\rvert$	33	25	36	31
$\lvert T^8\rvert$	27	20	34	27
$\lvert T^9\rvert$	21	18	33	25
$\lvert T^{10}\rvert$	18	16	33	21
$\lvert T^{11}\rvert$	18	13		18
$\lvert T^{12}\rvert$		8		14
$\lvert T^{13}\rvert$		4		10
$\lvert T^{14}\rvert$		2		8
$\lvert T^{15}\rvert$		1		6
$\lvert T^{16}\rvert$		0		4
$\lvert T^{17}\rvert$				2
$\lvert T^{18}\rvert$				1
$\lvert T^{19}\rvert$				0

Table 1: Testing the never-covered property on Subject 201.

0.4, and through further interpolation we obtain $\kappa^{**}=0.3229$.

Similar calculations are carried out for the other 46 subjects. The cumulative distributions of κ^* and κ^{**} are depicted in Figure 1. For each $r\in[0,1]$, we plot the percentage of subjects whose perturbation indices are less than or equal to r. Since for each subject $\kappa^*\leq\kappa^{**}$, the distribution of κ^{**} first-order stochastically dominates that of κ^*.

How does the perturbation index compare with other measures of a data set's rationality and, in particular, Afriat's critical cost efficiency index? Since a larger critical cost efficiency index is closer to rationality whereas a smaller perturbation index is closer to rationality, the two indices are naturally negatively correlated. In Table 2, we report the correlation coefficients between these two indices, making use of the critical cost efficiency indices calculated for the same set of subjects in

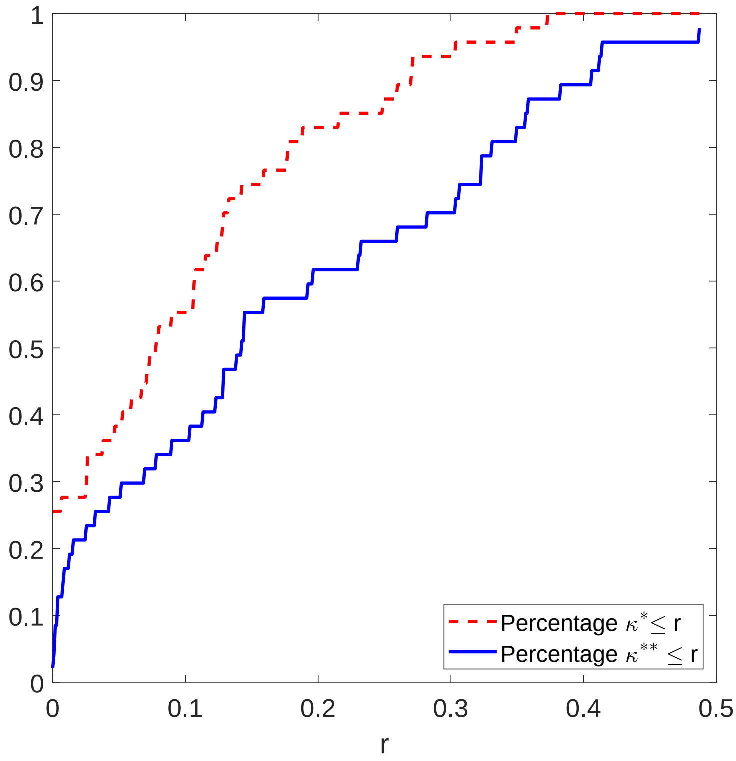

Figure 1: Distributions of κ^* and κ^{**}.

Polisson et al. (2020). Note that the indices under $\geq_{sym}$ are slightly less correlated than under $\geq$. In each case, the rank correlation coefficient is higher than the linear correlation coefficient. Lastly, the two indices are not perfectly correlated in any of the four cases, which suggests that how subjects perform on the perturbation index could convey information not conveyed by the critical cost efficiency index. It would be interesting to see how this index performs as a measure of rationality, compared to other measures; for example, whether, like the critical cost efficiency index, it can help explain broader economic outcomes (see Choi et al. (2014)). These are interesting topics for future study.

Linear Correlation Coefficient		Rank Correlation Coefficient	
$\trianglerighteq = \geq$	$\trianglerighteq = \geq_{sym}$	$\trianglerighteq = \geq$	$\trianglerighteq = \geq_{sym}$
-0.79499	-0.77732	-0.91911	-0.86829

Table 2: Correlation between perturbation index and critical cost efficiency index.

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