Rationalizing Choices in a Rich Domain

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Abstract

Two alternatives are independent if they never induce menu effects on each other. We propose a condition, Comparative Richness, which postulates that for every menu and every submenu of it, there is an alternative that is equally desirable as the submenu and independent of each alternative in the menu. We show that under Comparative Richness, (i) some axioms necessary for rationality become sufficient; (ii) several classic axioms without clear implications now characterize new choice models; (iii) interpretable specifications can be derived for existing models; and (iv) departures from rationality can be explicitly quantified.

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1 Introduction

Rationality, a fundamental assumption of economics, posits that a decision maker (DM) maintains a well-behaved preference ranking over choice alternatives irrespective of the choice menu she faces, and when encountering any menu she chooses the most preferred alternatives. Despite its normative appeal, empirical evidence over the last half-century has documented various violations of this assumption.[1] In particular, a recurring theme is that observed choices often exhibit menu effects—i.e., the relative desirability of alternatives often depends on the menu encountered.

In standard economic theory, rationality is jointly characterized by Sen's α and β (Sen, 1971).[2] Sen's α states that every chosen alternative in a larger menu that is available in a smaller menu should be chosen in the smaller menu; Sen's β states that if a chosen alternative in a smaller menu is also chosen in a larger menu, then any chosen alternative in the smaller menu should be chosen in the larger menu. Together, Sen's α and β preclude the possibility of menu effects, and thus ensure that the DM's choices from doubletons induce choices from larger menus in a consistent manner.

Conversely, a violation of either Sen's α or β invariably amounts to the situation in which the addition of an alternative (y) to a menu (A) increases the desirability of an alternative (x) relative to some other alternative in the menu. Specifically, if Sen's α is violated, we can always find x, y, and A such that x is not chosen in A but becomes chosen after the inclusion of y. In this scenario, the desirability of x relative to other alternatives in A is elevated by y. Similarly, if Sen's β is violated, there exist x, y, and A such that some other alternative z is chosen in A but is no longer chosen after the inclusion of y, whereas x is always chosen. In this scenario, the desirability of x relative to z is boosted by y.

The above observations suggest a natural way of identifying a trace of rationality from choices: Say that x and y are *independent* if they never induce menu effects on each other—i.e., the addition of one does not boost the desirability of the other relative to the rest of the alternatives in any given menu. If we restrict our attention to a subset of the choice domain in which the alternatives are pairwise independent, the DM will appear to be rational: Choices from doubletons reveal a preference ranking that pins down choices

[1] See, for instance, Tversky (1969); Grether and Plott (1979); Huber, Payne and Puto (1982); Simonson (1989); and Dembo, Kariv, Polisson and Quah (2021).

[2] Sen's α first appears as Postulate 4 in Chernoff (1954).

from larger menus in the standard manner. Even for a DM who systematically violates rationality, this preference ranking over independent alternatives may function as the cornerstone for identifying the logic behind the DM's choices.

In this paper, we offer a unified approach to study departures from rationality based on the aforementioned independence relation. We consider a rich choice domain in which for every choice menu, we can always find an alternative that is independent of every alternative in the menu. Under our key assumption, *Comparative Richness*, these independent alternatives help gauge the desirability of individual alternatives as well as menus of alternatives. In particular, a well-behaved preference ranking over *all menus*, which we refer to as the *shadow preference*, can be uniquely identified from the DM's choices. With Comparative Richness (and the resulting shadow preference), we investigate a variety of choice axioms that allow departures from rationality and obtain new models and characterization results.

Formally, Comparative Richness states that for all menus A, B with $B \subseteq A$, there exists an alternative x that is independent of every alternative in A and is as desirable as B in the sense that the chosen alternatives in $B \cup \{x\}$ include both x and some alternative in B.

To demonstrate Comparative Richness and how the shadow preference can be identified, consider a diner contemplating dessert options in a set course. Four options may be offered: cheesecake (C), rice pudding (P), plain yogurt (Y), and no dessert but a discount on the bill (\$). The diner is completely indifferent toward having any dessert or getting \$, with the exception that if *all* three desserts are available, regardless of the availability of \$, P stands out as the option that is not too unhealthy and reasonably tasty. In this example, \$ is independent of every dessert, since the availability of \$ does not boost the desirability of any dessert relative to other options, and vice versa. Furthermore, \$ is as desirable as any proper submenu of $\{C, P, Y\}$, since the DM is completely indifferent when offered each of these submenus of desserts together with \$.

Observe that the above example does not satisfy Comparative Richness: There does not exist an alternative that is independent of every dessert and is as desirable as $\{C, P, Y\}$. However, Comparative Richness can be satisfied if we add another option to the choice scenario. Suppose that a large discount (\$\$) may also be offered by the restaurant. Assume further that the diner makes exactly the same choices as before when \$\$ is not available; when \$\$ is available, the diner exclusively chooses \$\$ unless all three desserts are also available, in which case her choices will be $\{P, \$\$\}$. Now it can be verified that \$\$ is independent of any alternative (including itself trivially), and so is \$. Moreover, any menu that contains all three desserts or \$\$ will be as desirable as \$\$, and any other menu will be as desirable as \$, and thus Comparative Richness is established. Since \$\$ is naturally more desirable than \$, we obtain a ranking of desirability over all menus—i.e., the shadow preference—in the extended domain.

In fact, for any choice correspondence over a given choice domain, we can expand the domain and extend the choice correspondence properly, such that the extended choice correspondence satisfies Comparative Richness. Hence, Comparative Richness *alone* has minimal behavioral content. While Comparative Richness itself has little bite, if imposed together with additional axioms it strengthens many of them and delivers new behavioral implications.

First, under Comparative Richness, many axioms that are necessary for rationality also become sufficient. We find that under Comparative Richness, Sen's α, Sen's β, a weaker version of α, and a weaker version of β are all equivalent. Each of the four axioms holds if and only if the DM ranks every chosen alternative from a menu as desirable as the menu itself according to her shadow preference, which can further be shown to be equivalent to rationality. Thus, observed violations of one of these axioms will inevitably lead to violations of the other axioms as the choice domain becomes richer. Notably, the two weaker axioms are separately satisfied by many classic choice models, which indicates that those models intersect with Comparative Richness at rationality.

Second, under Comparative Richness, many axioms that do not have clear implications individually can now characterize interpretable choice models that appear to be new to the literature. We show that under Comparative Richness, Aizerman's axiom (Chernoff, 1954; Aizerman and Malishevski, 1981; Aizerman, 1985), which states that deleting unchosen alternatives does not affect choices, characterizes *MP-rationalizability*.[3] That is, there is a linear order over all menus such that given every menu A, the set of chosen alternatives maximizes the linear order over all submenus of A. The linear order can be constructed from the DM's shadow preference by keeping its asymmetric part and breaking ties properly. We also examine the Weaker Axiom of Revealed Preference (WrARP) (Jamison and Lau, 1973; Fishburn, 1975), which states that if x is revealed to be *strictly* better than y, then y cannot be revealed to be *strictly* better than x. It turns out

[3] "MP" stands for "menu preference."

that under Comparative Richness, WrARP characterizes *MO-rationalizability*,[4] which posits that the DM may either evaluate an alternative correctly or make a mistake by slightly overevaluating it. In this model, the DM's normative preference over alternatives can be constructed exactly as the restriction of her shadow preference to singleton menus.

Third, we demonstrate how Comparative Richness can provide additional insights into behavioral characterizations that are already valid even without it. We consider a natural generalization of the Weak Axiom of Revealed Preference with Limited Attention (Masatlioglu, Nakajima and Ozbay, 2012) to choice correspondences. We show that our axiom characterizes *GA-rationalizability*,[5] which posits that the DM pays attention to a subset of the feasible alternatives and chooses the ones that are more desirable than a menu-dependent benchmark. With Comparative Richness, however, we may construct a linear order over all menus based on the shadow preference such that within every menu A, the set of alternatives that catch the DM's attention maximizes the linear order over all submenus of A. Basically, Comparative Richness delivers a ranking of how "eye-catching" each set of alternatives is, which provides a story behind the attention filter in models with limited attention.

Last, we propose a new axiom, T-Weak Sen's α, which explicitly quantifies the DM's departure from rationality. T-Weak Sen's α states that for every alternative x, there are at most T distinct alternatives that can trigger violations of Sen's α on the choice of x. When $T = 1$, the axiom represents the weakest departure from Sen's α. We find that under Comparative Richness, T-Weak Sen's α characterizes $(T+1)$-*rationalizability*, which posits that the DM's evaluation of each alternative depends on at most T other feasible alternatives. When $T = 1$, $(T + 1)$-rationalizability nests the class of nontransitive preferences studied by Bell (1982); Loomes and Sugden (1982); Fishburn (1989); and Bordalo, Gennaioli and Shleifer (2012): There exists a function v such that x is preferred to y if and only if $v(x, y) \geq v(y, x)$. We show that $(T + 1)$-rationalizability can shed light on choices over lotteries with nontransitive preferences.

To demonstrate the broad applicability of our approach, we introduce several canonical examples in which Comparative Richness holds. We examine these examples in detail and offer additional ones in Section 3.2.

Rational choices. If the DM's choices satisfy rationality, then every alternative

[4] "MO" stands for "monotone overevaluation."
[5] "GA" stands for "generalized choices with limited attention."

is independent of itself as well as all other ones. For all menus A and B with $B \subseteq A$, we can simply pick an alternative x from the chosen ones in B. Then the chosen alternatives in $B \cup \{x\} = B$ automatically include both x and some alternative (x) in B. Thus, Comparative Richness holds.

Choice with monetary alternatives. Let the choice domain be $X \cup \mathbb{R}$, in which X is an arbitrary set of nonmonetary alternatives and each real number in $\mathbb{R}$ is interpreted as a monetary alternative. Assume that monetary alternatives cannot boost the desirability of any alternative relative to other options in any menu, and vice versa. Then Comparative Richness is satisfied if given every menu of nonmonetary alternatives B, there is a monetary alternative x such that the chosen alternatives in $B \cup \{x\}$ include both x and some alternative in B. Note that this requirement, together with the independence of monetary alternatives, imposes no restriction on the DM's choices from menus of nonmonetary alternatives. Basically, we only need the monetary alternatives to serve as the numeraire for desirability, as they do in the diner example.

Choice with attraction effects. Let the choice domain be $\mathbb{R}^n$, in which an alternative $x = (x^1, x^2, \cdots, x^n)$ is interpreted as a product that has quality x^i in attribute i. Suppose that the DM has the normative utility v that is continuous, strictly increasing in each argument, and for all i and x^{-i}, the range of $v(\cdot, x^{-i})$ is $\mathbb{R}$. Let v^+ be another utility function such that for every alternative x, $v^+(x) \geq v(x)$. Assume that the DM makes choices by maximizing the menu-dependent utility u: If there is $y \in A$ such that $x \geq y$ and $x \neq y$, then $u(x, A) = v^+(x)$; otherwise, $u(x, A) = v(x)$. In other words, the DM overevaluates x if there is a feasible alternative y that is dominated by x, which captures the attraction effect studied by Huber, Payne and Puto (1982); Huber and Puto (1983); and Ok, Ortoleva and Riella (2015). In this setting, Comparative Richness will always be satisfied.

Our paper is related to a large stream of the literature on choice models that generalize rationality. It has been shown that DMs may deviate from rational choices due to the adoption of multiple rationales (Manzini and Mariotti, 2007; Kalai, Rubinstein and Spiegler, 2002; Cherepavov, Feddersen and Sandroni, 2013); the status quo effect (Masatlioglu and Ok, 2005, 2014); limited memory and attention (Masatlioglu, Nakajima and Ozbay, 2012; Lleras, Masatlioglu, Nakajima and Ozbay, 2017; Bordalo, Gennaioli and Shleifer, 2020); reference dependence (Kőszegi and Rabin, 2006; Ok, Ortoleva and Riella, 2015); or framing and salience (Salant and Rubinstein, 2008; Bordalo, Gennaioli and Shleifer, 2012, 2013; Ellis and Masatlioglu, 2022). In contrast, we consider an independence relation that is revealed from the absence of menu effects between two alternatives and systematically study various kinds of departures from rationality under a richness condition on this relation. We characterize a series of models that appear to be new to the literature under this richness condition.

Our paper is also related to Richter (2020), as both papers derive results that connect individual choices with preferences over menus. In particular, the *choose twice* procedure proposed in Richter (2020) involves a DM who has a linear order over sets and a linear order over alternatives. Given a menu, she first chooses the most preferred submenu, then chooses the most preferred alternative from the submenu. Our notion of GA-rationalizability with the consideration set determined by a linear order over menus is a generalization of the choose twice procedure to choice correspondences. We elaborate on the connection in Section 4.2.3.

This is not the first paper in which the richness of the DM's choices plays an important role. In subjective expected utility theory, the fineness condition of Savage (1954) posits that the state space can be divided into an arbitrary number of events, each with a sufficiently small chance of occurrence. In the context of random choices, Gul, Natenzon and Pesendorfer (2014) build on a richness condition that is similar in spirit to Comparative Richness, in the sense that both conditions alone have minimal behavioral implications. Comparative Richness distinguishes itself from these two conditions by allowing the choice domain to be finite. In this sense, Comparative Richness *does* bear empirical relevance.

To avoid redundancy, we discuss other related work in each section. We organize the paper as follows. We present baseline notation in Section 2 and our notion of independence and Comparative Richness in Section 3. In Section 4, we study the classic choice axioms under Comparative Richness, and in Section 5 we introduce T-weak Sen's α and study its implications. The Appendix contains all proofs omitted from the main body of the paper.

2 Preliminaries

For any nonempty set H, a *binary relation* $\succsim$ over H is a subset of $H \times H$. For any $a, b \in H$, write $a \succsim b$ if $(a, b) \in \succsim$ and $a \not\succsim b$ if $(a, b) \notin \succsim$. A binary relation

$\succsim$ is *reflexive* if for all $a \in H$, $a \succsim a$; *complete* if for all $a, b \in H$, either $a \succsim b$ or $b \succsim a$; *transitive* if $a \succsim b$ and $b \succsim c$ implies $a \succsim c$; *asymmetric* if $a \succsim b$ implies $b \not\succsim a$; and *antisymmetric* if $a \succsim b$ and $b \succsim a$ implies $a = b$. A complete and transitive binary relation is called a *preference*, and an antisymmetric preference is called a *linear order*. For a given binary relation $\succsim$, we denote by $\succ$ its asymmetric part and $\sim$ its symmetric part, i.e., $a \succ b$ if and only if $a \succsim b$ and $b \not\succsim a$; $a \sim b$ if and only if $a \succsim b$ and $b \succsim a$. We also use $\succ$ to denote an asymmetric binary relation when there is no risk of confusion. For any two binary relations $\succsim_1$ and $\succsim_2$, we say that $\succsim_1$ *extends* $\succsim_2$ if for all $a, b \in H$, $a \succsim_1 b$ implies $a \succsim_2 b$, and $a \succ_1 b$ implies $a \succ_2 b$.

For a given binary relation $\succsim$ over H and a finite subset $K \subseteq H$, we denote by $\mathcal{U}_{\succsim}(K)$ the elements in K that are undominated with respect to the asymmetric part of $\succsim$, i.e., $\mathcal{U}_{\succsim}(K) = \{a \in K : \forall b \in K, \, b \not\succ a\}$. When $\succsim$ is complete, $\mathcal{U}_{\succsim}(K)$ contains alternatives in K that are ranked higher than any other element in K with respect to $\succsim$—i.e., $\mathcal{U}_{\succsim}(K) = \{a \in K : \forall b \in K, a \succsim b\}$.

For a given nonempty set of alternatives X, a *menu* is a nonempty and finite subset of it. For any nonempty $Y \subseteq X$, we use $\mathcal{M}(Y)$ to denote the collection of all menus of X that are contained in Y. We use A, B, C to denote generic menus and x, y, z to denote generic alternatives. For a given set of alternatives X and a mapping $f : \mathcal{M}(X) \to \mathcal{M}(X)$, the tuple (X, f) is said to be a *choice correspondence* if for all $A \in \mathcal{M}(X)$, $f(A) \subseteq A$. When there is no risk of confusion regarding the choice space (usually denoted by X), we will also refer to f as a choice correspondence. For any two choice correspondences (X, f) and (X', f'), we say that (X', f') is an *extension* of (X, f) if $X \subseteq X'$ and for all $A \in \mathcal{M}(X)$, $f(A) = f'(A)$. We say that a preference $\succsim$ over X *rationalizes* a choice correspondence f if for all $A \in \mathcal{M}(X)$, $f(A) = \mathcal{U}_{\succsim}(A)$. A choice correspondence (X, f) is *rationalizable* if it can be rationalized by a preference over X.

3 Independence and Comparative Richness

In this section, we introduce a new notion of independence to capture the situation in which two alternatives never induce menu effects on each other. Then we introduce our richness condition based on this independence relation.

3.1 Independence

To motivate our notion of independence, we first present the benchmark axioms that characterize rationalizable choice correspondences.

Axiom 1 (Sen's α)**.** *For all $x \in X$ and $A, B \in \mathcal{M}(X)$ with $x \in B \subseteq A$, $x \in f(A)$ implies $x \in f(B)$.*

Axiom 2 (Sen's β)**.** *For all $x \in X$ and $A, B \in \mathcal{M}(X)$ with $x \in B \subseteq A$, if $x \in f(A) \cap f(B)$, then $f(B) \subseteq f(A)$.*

It is well known that a choice correspondence is rationalizable if and only if it satisfies Sen's α and β. As discussed in the Introduction, an underlying principle of the two axioms is that the addition of one alternative cannot boost the desirability of one feasible alternative relative to other feasible ones. Our definition of independence is motivated by this observation. Given a choice correspondence f, for any two alternatives x and y, we say that x and y are *independent*, or x is *independent of* y, denoted by $x \perp y$, if for every $A \in \mathcal{M}(X)$:[6]

$$x \in f(A \cup \{y\}) \cap A \Rightarrow x \in f(A) \subseteq f(A \cup \{y\}),$$

$$y \in f(A \cup \{x\}) \cap A \Rightarrow y \in f(A) \subseteq f(A \cup \{x\}).$$

By the definition, each alternative is independent of itself. To demonstrate the definition, consider $x \neq y$: If x is chosen in $A \cup \{y\}$ but not in A—a violation of Sen's α—then the addition of y must boost the desirability of x relative to chosen alternatives in A; if x is chosen in both A and $A \cup \{y\}$ but $f(A) \not\subseteq f(A \cup \{y\})$—a violation of Sen's β—then the addition of y must boost the desirability of x relative to the alternatives in $f(A) \backslash f(A \cup \{y\})$. Thus, the independence between x and y implies that for any menu that contains x, adding y to the menu does not boost the desirability of x relative to any other alternative in the menu, and vice versa. The following proposition demonstrates how independence relates to rationalizability. Its proof is evident and thus omitted.

Proposition 1. *A choice correspondence (X, f) is rationalizable if and only if for all $x, y \in X$, $x \perp y$.*

Two *sets* A and B are said to be independent, denoted by $A \perp B$, if for all $x \in A$ and $y \in B$, $x \perp y$.[7] A collection of menus $\{A_i\}_{i \in I}$ is said to be a *mutually*

[6] More rigorously, we should write $\perp_f$ to indicate that the independence relation depends on the underlying choice correspondence f. We write $\perp$ instead of $\perp_f$ whenever there is no risk of confusion.

[7] With a harmless abuse of notation, we write $x \perp A$ instead of $\{x\} \perp A$.

independent collection (MIC) if for all $i, j \in I$ with $i \neq j$, we have $A_i \perp A_j$. We will refer to each element of an MIC as a *block*.[8]

Theorem 1. *Let $\{A_i\}_{i\in I}$ be an MIC. For all $i, j \in I$ and menus A, B such that $A_i \cup A_j \subseteq B \subseteq A \subseteq \bigcup_{k\in I} A_k$, the following statements hold:*

(1) $f(A) \cap A_i \in \{f(A_i), \emptyset\}$;

(2) *If* $f(A) \cap A_i \neq \emptyset$, *then* $f(B) \cap A_i \neq \emptyset$;

(3) *If* $f(B) \cap A_i \neq \emptyset$, $f(B) \cap A_j \neq \emptyset$ *and* $f(A) \cap A_i \neq \emptyset$, *then* $f(A) \cap A_j \neq \emptyset$.

Statement (1) states that for any given block, the set of alternatives within the block that gets chosen, if nonempty, must be the same in all menus that contain the block. If we view each block of the MIC as an "alternative," then statements (2) and (3) can be interpreted as the counterpart of Sen's α and β, respectively. The following proposition makes the analogy clear.

Proposition 2. *Let $\Pi = \{A_i\}_{i\in I}$ be an MIC. There exists a unique preference $\trianglerighteq$ over Π such that for every $i \in I$ and every nonempty and finite $J \subseteq I$, $f(\bigcup_{j\in J} A_j) \cap A_i \neq \emptyset$ if and only if $A_i \in \mathcal{U}_{\trianglerighteq}(\{A_j\}_{j\in J})$.*

For any Π that is an MIC, the preference $\trianglerighteq$ over Π that satisfies the condition of Proposition 2 is said to *rationalize f on* Π. Theorem 1 and Proposition 2 together suggest that when the DM faces a menu that is the union of an MIC, her choices can be modeled as a two-stage process: First, she selects the most preferred blocks according to a preference over the blocks; second, she makes choices from each selected block.

Proposition 2 also implies the following corollary, which states that if two MICs contain common blocks, then the preferences that rationalize f on the two MICs must agree on the rankings among the common blocks. The proofs of Proposition 2 and the following corollary are trivial and thus omitted.

Corollary 1. *For all MICs Π_1 and Π_2, if $A, B \in \Pi_1 \cap \Pi_2$, then $A \trianglerighteq_1 B$ if and only if $A \trianglerighteq_2 B$, where $\trianglerighteq_1$ and $\trianglerighteq_2$ rationalize f on Π_1 and Π_2, respectively.*

Define a binary relation $\trianglerighteq_0$ over $\mathcal{M}(X)$ such that $A \trianglerighteq_0 B$ if and only if $A \perp B$ and $f(A \cup B) \cap A \neq \emptyset$. Note that the restriction of $\trianglerighteq_0$ to any MIC is a preference, but $\trianglerighteq_0$ itself may not be a preference: Any two menus that are not independent cannot be ranked by $\trianglerighteq_0$. Corollary 1 implies that this binary relation governs the DM's choices over blocks whenever she faces a menu that

[8] Note that it is not necessarily true that two blocks of an MIC are disjoint.

is the union of an MIC. Thus, we interpret this binary relation as a ranking of desirability over menus. Notably, when ranking any pair of menus, $\trianglerighteq_0$ takes into account all menu effects among alternatives within each menu but excludes any menu effects between alternatives from different menus. As will be shown in the next section, Comparative Richness offers a natural way of extending this desirability ranking to all menus.

3.2 Comparative Richness

We proceed to introduce our richness condition. Given a choice correspondence f, we say that an alternative x is *comparable to* a menu A *under* f, denoted as $x \approx^f A$, if $x \in f(A \cup \{x\})$ and $f(A \cup \{x\}) \cap A \neq \emptyset$. In other words, x is comparable to A if both x and some alternative in A are chosen in $A \cup \{x\}$. Note that $x \in f(A)$ directly implies $x \approx^f A$.

Definition 1. *A choice correspondence (X, f) satisfies Comparative Richness if for all menus A and B with $B \subseteq A$, there exists $x \in X$ such that $x \perp A$ and $x \approx^f B$.*

Comparative Richness states that for any menu A and any submenu B of A, we can find an alternative x that is independent of A and comparable to B. With this condition, we can compare the desirability of any two menus via independent alternatives, regardless of whether they are independent. To see this, consider two arbitrary menus A' and B'. Comparative Richness enables us to find x and y such that $x \perp y$, $\{x, y\} \perp A' \cup B'$, $x \approx^f A'$, and $y \approx^f B'$.[9] It follows that $\{x\}$ and $\{y\}$ are as desirable as A' and B', respectively.[10] Since $x \perp y$, $\{x\}$ is more desirable than $\{y\}$ if and only if x is chosen in $\{x, y\}$. Thus, the comparison of the desirability of A' and B' can be revealed by the choice made in $\{x, y\}$. We formalize our analysis in the following definition.

Definition 2. *Given a choice correspondence (X, f), the shadow order of f, denoted by $\trianglerighteq^f$, is a binary relation over $\mathcal{M}(X)$ such that for all menus A and B, $A \trianglerighteq^f B$ if there exist $x, y \in X$ such that $x \perp y$, $\{x, y\} \perp A \cup B$, $x \approx^f A$, $y \approx^f B$,*

[9] To see how Comparative Richness implies the existence of such x and y, note that we can first pick $x \in X$ such that $x \perp A' \cup B'$ and $x \approx^f A'$. Then we pick $y \in X$ such that $y \perp A' \cup B' \cup \{x\}$ and $y \approx^f B'$.

[10] To clarify the language, the desirability of alternative x may depend on the set of feasible alternatives, but the desirability of the menu $\{x\}$ excludes any menu effects that other alternatives may induce on x.

and $x \in f(\{x,y\})$. *We use* $\rhd^f$ *and* $\bowtie^f$ *to denote the asymmetric and symmetric parts of* $\unrhd^f$*, respectively.*

Our next theorem demonstrates that Comparative Richness enables us to identify a unique desirability ranking over all menus through the procedure described above.

Theorem 2. *A choice correspondence* (X, f) *satisfies Comparative Richness if and only if the shadow order* $\unrhd^f$ *is a preference. Furthermore, if* (X, f) *satisfies Comparative Richness, the following statements hold:*

(1) *For all menus* A *and* B, $A \unrhd^f B$ *if and only if for all* $x, y \in X$ *with* $x \perp y$, $\{x,y\} \perp A \cup B$, $x \approx^f A$ *and* $y \approx^f B$, *we have* $x \in f(\{x,y\})$*;*

(2) *The shadow order* $\unrhd^f$ *rationalizes* f *on every MIC;*

(3) *For all menus* A *and* B, *if* $B \subseteq A$, *then* $A \unrhd^f B$*; if* $B \subseteq A$ *and* $B \unrhd^f A$, *then* $f(B) \subseteq f(A)$.

By Theorem 2, Comparative Richness is not only sufficient but also necessary for the shadow order to be a preference. If a choice correspondence f satisfies Comparative Richness, we will refer to $\unrhd^f$ as the *shadow preference* of f. We interpret $\unrhd^f$ as a ranking of desirability over all menus. In particular, statement (2) implies that $\unrhd^f$ is an extension of the desirability ranking $\unrhd_0$ that we define for mutually independent menus. Since $\unrhd^f$ is transitive and is obtained from $\unrhd_0$ through indirect comparison, $\unrhd^f$ also coincides with the transitive closure of $\unrhd_0$.[11] Thus, Comparative Richness implies that the transitive closure of $\unrhd_0$ extends $\unrhd_0$ and is complete.

Statement (3) states that larger menus are more desirable, and choices made in a smaller menu remain chosen in a larger one when the two menus are equally desirable. We will refer to this property as the *monotonicity* of the shadow preference. To see why monotonicity holds, let $A = B \cup \{x\}$ and assume that there is an alternative y that is independent of A and comparable to B. Applying statement (2) to the MIC $\{B, \{y\}\}$ yields $B \bowtie^f \{y\}$. Thus, we can use y as a benchmark for comparing the desirability of A and B. To see why larger menus are more desirable, consider the MIC $\{A, \{y\}\}$ and suppose to the contrary that $\{y\} \rhd^f A$. This means that only y is chosen in the menu $A \cup \{y\}$ and $y \notin A$. However, it follows that the addition of x to $B \cup \{y\}$ has increased the

[11]Formally, for any given binary relation $\succsim$ over H, its transitive closure is a binary relation $\succsim^+$ over H such that for all $a, b \in H$, $a \succsim^+ b$ if there is a finite sequence $(a_k)_{k=1}^n$ such that $a_1 = a$, $a_n = b$, and for all $k \in \{1, ..., n-1\}$, $a_{k+1} \succsim a_k$.

desirability of y relative to other chosen alternatives in B, which cannot be the case since $x \perp y$. Thus, we conclude that $A \trianglerighteq^f \{y\} \bowtie^f B$. To see the second part of monotonicity, assume further that $A \bowtie^f \{y\} \bowtie^f B$. Since $y \perp A$, we have $f(A \cup \{y\}) = f(A) \cup \{y\}$ and $f(B \cup \{y\}) = f(B) \cup \{y\}$. If there exists $z \in f(B)$ but $z \notin f(A)$, then the fact that y is chosen in both $A \cup \{y\}$ and $B \cup \{y\}$ implies that the addition of x to the menu $B \cup \{y\}$ has boosted the desirability of y relative to z, which cannot happen since $x \perp y$.

To illustrate Theorem 2, we revisit the examples presented in the Introduction and provide an additional example.

Rational choices revisited. Let (X, f) be rationalized by a preference $\succsim$ over X. For any menus A and B with $B \subseteq A$, pick $x \in \mathcal{U}_{\succsim}(B)$. We have $x \approx^f B$ and $x \perp A$. Thus, Comparative Richness holds. The shadow preference $\trianglerighteq^f$ satisfies that for all menus A and B, $A \trianglerighteq^f B$ if and only if there is $x \in A$ such that for all $y \in B$, $x \succsim y$.

Choice with monetary alternatives revisited. Let $X \cup \mathbb{R}$ be the choice domain. Let u be a menu-dependent utility function such that for all menu A and $x \in A$, $u(x, A) = x$ if $x \in \mathbb{R}$, and $u(x, A) = u(x, A \backslash \mathbb{R})$ if $x \in X$. For all menu A and $x, y \in X$, we require $u(x, A) \leq u(x, A \cup \{y\})$ to allow for the possibility that nonmonetary alternatives may boost the desirability of each other relative to the monetary alternatives. Let the choice correspondence f be such that for every menu A, $f(A) = \arg\max_{x \in A} u(x, A)$.

We show that for all alternatives x and y, if $x \in \mathbb{R}$, then $x \perp y$. To see this, first consider adding the monetary alternative x to a menu A that contains y. Since the utility of every alternative in A is unaffected by the addition of x, if y is chosen in $A \cup \{x\}$, then y must also be chosen in A, and every chosen alternative in A, which has the same menu-dependent utility as y in both A and $A \cup \{x\}$, will also be chosen in $A \cup \{x\}$. Now, consider adding alternative y to menu B that contains x. Suppose that x is chosen in $B \cup \{y\}$. Since the utility of x is always equal to x and the utility of any $z \in B$ can only increase with the addition of y, it must be the case that x is chosen in B. Furthermore, consider any chosen alternative w in B. The utility of w in B must be exactly x. Since the addition of y cannot decrease the utility of w, the utility of w in $B \cup \{y\}$ cannot be lower than x. Then the fact that x is chosen in $B \cup \{y\}$ implies that w must also be chosen in $B \cup \{y\}$.

By the above argument, Comparative Richness holds since for any two menus A and B with $B \subseteq A$, there exists $x \in \mathbb{R}$ such that $\max_{y \in B} u(y, B) = x$, which ensures that x is comparable to B. The shadow preference then satisfies that for all menus A and B, $A \trianglerighteq^f B$ if and only if $\max_{x\in A} u(x,A) \geq \max_{y\in B} u(y,B)$.

Choice with attraction effects revisited. Let $(\mathbb{R}^n, f)$ be the choice correspondence such that for every menu A, $f(A) = \arg\max_{x\in A} u(x,A)$, where the definition of u follows from the Introduction. Since the desirability of an alternative can only be boosted by a dominated alternative, following the argument in the previous example, we can show that two alternatives are independent if they do not dominate each other. For any two menus A and B with $B \subsetneq A$, we can find an alternative $x = (x^1, ..., x^n)$ such that $x \perp A$ and $v(x) = \max_{y\in B} u(y,B)$, where the non-domination relation between x and alternatives in A can be guaranteed by setting x^1 to be large and x^2 to be small. Clearly, $u(x, B\cup\{x\}) = v(x) = \max_{y\in B} u(y,B) = \max_{y\in B} u(y, B\cup\{x\})$, which indicates that x is comparable to B. Therefore, Comparative Richness holds. The shadow preference satisfies that for all menus A and B, $A \trianglerighteq^f B$ if and only if $\max_{x\in A} u(x,A) \geq \max_{y\in B} u(y,B)$.

Bounded context-dependent preferences. Let $X = \mathbb{R}^n$ $(n \geq 2)$ be the space of alternatives, in which each dimension represents a specific attribute. Each alternative $x \in X$ takes the form $x = (x^1, ..., x^n)$ with its coordinate in each dimension denoting its quality in that attribute. We consider the context-dependent preferences introduced by Tversky and Simonson (1993): For each $A \in \mathcal{M}(X)$ and $x \in A$, the menu-dependent utility of x in A is given by $u^*(x,A) = v(x) + \sum_{y\in A} C(x,y)$. The term $v(x)$ can be viewed as the normative utility of x, and $C(x,y)$ is the comparison utility of x when compared with some other alternative y in the menu. Let the choice correspondence f be defined such that for all $A \in \mathcal{M}(X)$, $f(A) = \arg\max_{x\in A} u^*(x,A)$.

We consider a specific functional form of the context-dependent preference model. Let $v(x) = \sum_{k=1}^n v^k(x^k)$, where $v^k : \mathbb{R} \to \mathbb{R}$ is monotone and onto. Define

$$\hat{C}(x,y) = \eta \sum_{k=1}^{n} (\max\{v^k(x^k) - v^k(y^k), 0\} + \lambda \min\{v^k(x^k) - v^k(y^k), 0\})$$

for some $\eta > 0$ and $\lambda > 1$, and let $C(x,y) = \max\{\hat{C}(x,y), L\}$ for some constant $L < 0$. The function $\hat{C}$ takes the same functional form as the reference-dependent utility in Kőszegi and Rabin (2006). To interpret, $\hat{C}(x,y)$ additively aggregates the advantages and disadvantages of x over y, with the disadvantages having a higher weight due to the loss aversion of the DM. The extra component of C is the lower bound L, which means that the evaluation of a given alternative cannot be reduced too much by the presence of another alternative.

We argue that for any $x, y \in X$, $x \perp y$ if and only if $C(x,y) = C(y,x) = L$. Clearly, if $C(x,y) = C(y,x) = L$, then the addition of x to any menu cannot boost the desirability of y relative to any other feasible alternative, and vice versa. Conversely, suppose $C(x,y) > L$ or $C(y,x) > L$. Since $v(x) \geq v(y)$ implies $C(x,y) \geq C(y,x)$, we can assume WLOG that $v(x) \geq v(y)$ and $C(x,y) > L$. Consider an alternative z such that $v(z) = v(x)$ and $C(x,z) = C(z,x) = C(y,z) = C(z,y) = L$ (the strategy to find such an alternative z is to make $v^1(z^1)$ large enough and $v^2(z^2)$ small enough without changing $v(z)$). Now, we have $u^*(x,\{x,z\}) = u^*(z,\{x,z\}) = v(x)+L$ and $u^*(x,\{x,y,z\}) = v(x)+L+C(x,y) > v(x)+2L = u^*(z,\{x,y,z\})$. Thus, x, z are chosen in $\{x,z\}$, but only x is chosen in $\{x,y,z\}$, which implies that x and y are not independent.

Note that the choice correspondence f can be equivalently induced by the maximization of the adjusted menu-dependent utility u such that for all menu A and $x \in A$, $u(x,A) = u^*(x,A) - (|A|-1)L$. With the above characterization of the independence relation, it can be shown that $x \perp y$ if and only if for all menus A and B with $x \in A$ and $y \in B$, $u(x,A) = u(x, A \cup \{y\})$ and $u(y,B) = u(y, B \cup \{x\})$. Consider menus A and B with $B \subseteq A$. By definition, $x \approx^f B$ if and only if $u(x, B \cup \{x\}) = \max_{y \in B} u(y, B \cup \{x\})$. If $x \perp A$, it is easy to see that $x \approx^f B$ if and only if $v(x) = u(x,\{x\}) = \max_{y \in B} u(y,B)$. To show that f satisfies Comparative Richness, it suffices to find $x \perp A$ with $v(x) = \max_{y \in B} u(y,B)$. To do so, we just need to find an x with the right $v(x)$ and then increase $v^1(x^1)$ and decrease $v^2(x^2)$ to ensure $x \perp A$ without varying $v(x)$. The shadow preference satisfies that for all menus A and B, $A \trianglerighteq^f B$ if and only if $\max_{x \in A} u(x,A) \geq \max_{y \in B} u(y,B)$.

3.3 Behavioral Implication of Richness

In this section, we show that Comparative Richness alone has minimal behavioral content. We show that for any choice correspondence over a given choice domain, we can expand the domain and extend the choice correspondence properly such that the extended choice correspondence satisfies Comparative Richness. Recall that for any two choice correspondences (X,f) and (Y,g), we say that (Y,g) is an extension of (X,f) if $X \subseteq Y$ and for all $A \in \mathcal{M}(X)$, $f(A) = g(A)$.

Definition 3. *Given a choice correspondence (X,f), a preference $\trianglerighteq$ over $\mathcal{M}(X)$ is a quasi-shadow preference of f if it satisfies monotonicity—i.e., for all menus A and B, if $B \subseteq A$, then $A \trianglerighteq B$; if $B \subseteq A$ and $B \trianglerighteq A$, then $f(B) \subseteq f(A)$. We* *use $\rhd$ and $\bowtie$ to denote its asymmetric and symmetric parts, respectively.*

Note that for any choice correspondence f, there exists a quasi-shadow preference of it: Consider the preference $\trianglerighteq$ over $\mathcal{M}(X)$ such that for all $A, B \in \mathcal{M}(X)$, $A \trianglerighteq B$ if and only if $|A| \geq |B|$, and it clearly satisfies monotonicity.

Our next proposition demonstrates that any choice correspondence admits an extension that satisfies Comparative Richness. Furthermore, we have the freedom to regulate the extension with an arbitrary quasi-shadow preference. Before presenting the proposition, we first define the canonical extensions of choice correspondences.

Definition 4. *Given a choice correspondence (X, f) and a quasi-shadow preference $\trianglerighteq$ of it, an extension (Y, g) of (X, f) is $\trianglerighteq$-canonical if there is a bijection $\Phi : Y \backslash X \to \mathcal{M}(X)$ such that for every $A \in \mathcal{M}(Y)$:*

$$g(A) = \begin{cases} f(A \cap X) \cup \{x \in A \backslash X : A \cap X \bowtie \Phi(x)\}, \text{ if } \forall x \in A \backslash X,\ A \cap X \trianglerighteq \Phi(x), \\ \{x \in A \backslash X : \forall y \in A \backslash X, \Phi(x) \trianglerighteq \Phi(y)\}, \quad \text{otherwise.} \end{cases}$$

The canonical extension (Y, g) is constructed such that each auxiliary alternative $y \in Y \backslash X$ corresponds to a menu $\Phi(y)$ in $\mathcal{M}(X)$ with the desirability of y being determined by the ranking of $\Phi(y)$ under $\trianglerighteq$ irrespective of the choice menu encountered. For any $A \in \mathcal{M}(Y)$, the extended choice correspondence compares the desirability of menu $A \cap X$ with that of each auxiliary alternative in $A \backslash X$. If $A \cap X$ is weakly more desirable than every auxiliary alternative, then $g(A)$ contains the choices made in menu $A \cap X$ according to the original choice correspondence f, together with the auxiliary alternatives that are as desirable as $A \cap X$. If some auxiliary alternative is strictly more desirable than $A \cap X$, then the most desirable auxiliary alternatives are selected by g.

Proposition 3. *Let (X, f) be an arbitrary choice correspondence and $\trianglerighteq$ an arbitrary quasi-shadow preference of it. If (Y, g) is a $\trianglerighteq$-canonical extension of (X, f), then (Y, g) satisfies Comparative Richness, and $\trianglerighteq^g$ coincides with $\trianglerighteq$ on $\mathcal{M}(X)$.*

In the proof of Proposition 3, we show that for any canonical extension (Y, g), the auxiliary alternatives are (i) mutually independent, (ii) independent of every alternative in X, and (iii) comparable to their corresponding menus. To see why we can accommodate any quasi-shadow preference when constructing the canonical extension, consider menus $A, B \subseteq X$ and suppose $A \perp B$ under the original choice correspondence (X, f). Let x and y be the auxiliary alternatives that correspond to A and B, respectively. On the one hand, if neither A nor B is a subset of the other, then we have complete freedom to implement any desirability ranking between A and B by specifying $g(\{x, y\})$ accordingly. Even if $g(\{x, y\})$ is not aligned with $g(A \cup B)$ (i.e., $f(A \cup B)$)—for example, $g(A \cup B) \cap A = \emptyset$ but $x \in g(\{x, y\})$, it just means that A and B are no longer independent under the extended choice correspondence. In other words, the independence relation under the original choice correspondence does not necessarily need to be preserved in the extension procedure. On the other hand, if $B \subseteq A$, we lose the freedom to specify $g(\{x, y\}) = \{y\}$ and set $B \rhd^g A$. To see why, suppose to the contrary that $g(\{x, y\}) = \{y\}$ and $B \rhd^g A$. It follows that $\{y\} \bowtie^g B \rhd^g A \bowtie^g \{x\}$, and thus $g(B \cup \{x\}) = g(B)$ and $g(A \cup \{x\}) = g(A) \cup \{x\}$. This means that the addition of some alternative in $A \backslash B$ has boosted the desirability of x relative to other feasible alternatives, which contradicts the fact that x is independent of A.

Notably, if a given choice correspondence already satisfies Comparative Richness, Proposition 3 implies that we can always extend the choice correspondence to a larger domain such that the shadow preference of the extended choice correspondence extends the original one.

4 Weak Rationality Meets Richness

In this section, we study classic choice axioms that are proposed in the literature to characterize rationality or boundedly rational choice models.

4.1 Characterizing Rationality

We explore axioms that characterize rationality under Comparative Richness—i.e., when Comparative Richness is satisfied, those axioms are necessary and sufficient for a given choice correspondence to be rationalizable. We start with weakenings of Sen's α and β.

Axiom 3 (Binary Sen's α)**.** *For all $A \in \mathcal{M}(X)$ and $x, y \in f(A)$, $f(\{x, y\}) = \{x, y\}$.*

Axiom 4 (Binary Sen's β)**.** *For all $A \in \mathcal{M}(X)$ and $x, y \in A$, if $f(\{x, y\}) = \{x, y\}$ and $x \in f(A)$, then $y \in f(A)$.*

Binary Sen's α states that if two alternatives x and y are both chosen in some menu A, then both are chosen in the binary menu $\{x, y\}$. Clearly, this axiom weakens Sen's α, since the choice of x and y in A can only imply their choice in the binary menu $\{x, y\}$ but not in every submenu of A that contains x and y. Binary Sen's β states that if two alternatives x and y are both chosen in the binary menu $\{x, y\}$, then they must be simultaneously chosen or unchosen in any given menu A that contains them.

To state the next axiom, we define Condorcet winners. Specifically, for any menu A, an alternative $x \in A$ is said to be a *Condorcet winner* in A if for all $y \in A$, we have $x \in f(\{x, y\})$. That is, x is a Condorcet winner if it is chosen when compared with every other alternative in the menu. Denote by $\mathcal{W}(A)$ the set of all Condorcet winners in menu A. The next axiom states that the set of choices made in a given menu should coincide with the set of Condorcet winners if the latter is nonempty.

Axiom 5 (Condorcet Consistency)**.** *For all $A \in \mathcal{M}(X)$, if $\mathcal{W}(A) \neq \emptyset$, then $f(A) = \mathcal{W}(A)$.*

The following theorem states that under Comparative Richness, Binary Sen's α, Binary Sen's β, and Condorcet Consistency all lead to rational choice behavior.

Theorem 3. *If a choice correspondence (X, f) satisfies Comparative Richness, then the following statements are equivalent:*

(1) *(X, f) satisfies Binary Sen's α;*

(2) *(X, f) satisfies Binary Sen's β;*

(3) *(X, f) satisfies Condorcet Consistency;*

(4) *(X, f) is rationalizable.*

(5) *For all menu A and $x \in X$, $x \in f(A)$ implies $\{x\} \bowtie^f A$.*

Thus, observed violations of any one of Sen's α, Sen's β, Binary Sen's α, Binary Sen's β, or Condorcet Consistency will inevitably lead to violations of the other four axioms as the choice domain becomes richer.

Notably, many boundedly rational choice models satisfy one of these axioms. Thus, Theorem 3 implies that these models intersect with Comparative Richness at rationality.

Choice of undominated alternatives. Jamison and Lau (1973, 1975) and Fishburn (1975) study undominated choices under a binary relation $\succsim$ such that for every menu A, $f(A) = \mathcal{U}_{\succsim}(A)$. It is well known that such a choice correspondence is well defined if and only if the asymmetric part $\succ$ of the binary relation is acyclic.[12] We note that such a choice correspondence f satisfies Sen's α: If x is selected in a menu A, then it is not dominated (with respect to $\succ$) by any other alternative in A and thus is chosen in any smaller menu that contains it. Hence, this model intersects with our richness condition at rationality.

Choice with a sequence of α filters. An important stream of the literature concerns DMs who conduct multiple rounds of eliminations of alternatives to reach their final choices. For instance, Manzini and Mariotti (2007) study choices with an ordered sequence of rationales (binary relations) whereby the DM eliminates dominated alternatives sequentially.[13] Cherepavov, Feddersen and Sandroni (2013) consider a DM who has a preference over the alternatives and a set of rationales. For any given menu, the DM first eliminates alternatives that are dominated under *every* rationale, and then selects her most preferred alternative according to her preference within the remaining alternatives.[14]

The choice models above can be generalized by a sequential procedure such that in every round, the DM applies a filter to eliminate alternatives. Specifically, an α *filter* is a function $\Gamma : \mathcal{M}(X) \to 2^X$ such that for all menus A and B, (i) $\Gamma(A) \subseteq A$, and (ii) $\Gamma(A \cup B) \cap B \subseteq \Gamma(B)$. We note that property (ii) is a restatement of Sen's α. We say that a choice correspondence f is *sequentially α-filtered* if there exists a finite sequence of α filters $(\Gamma_k)_{k=1}^n$ such that $f = \Gamma_n \circ \Gamma_{n-1} \circ \cdots \circ \Gamma_1$.

In fact, a choice correspondence satisfies Binary Sen's α if it is sequentially α-filtered.[15] To see this, consider $x, y \in f(A) = \Gamma_n \circ \Gamma_{n-1} \circ \cdots \circ \Gamma_1(A)$. Since x and y are not filtered out by any α filter in the sequence, we have that for each $k \in \{1, ..., n\}$, there exists $B_k \subseteq A$ such that $x, y \in \Gamma_k(B_k)$. Thus, for each k, $\Gamma_k(\{x, y\}) = \{x, y\}$ and we have $f(\{x, y\}) = \{x, y\}$.

To summarize, any choice model in which choices are sequentially α-filtered, including those by Manzini and Mariotti (2007) and Cherepavov, Feddersen and Sandroni (2013), intersects with our richness condition at rationality.

Top-cycle choices. A large stream of the literature has investigated choices

[12] The binary relation $\succ$ is said to be acyclic if for any finite sequence of alternatives $(x_i)_{i=1}^{n+1}$ with $x_1 = x_{n+1}$, it cannot happen that for all $k \in \{1, ..., n\}$, $x_k \succ x_{k+1}$.

[13] See also Au and Kawai (2011) for rationalizing sequential choices with transitive rationales.

[14] See also Ridout (2021) for the situation in which each rationale is a preference. Ridout (2021) also offers an axiomatic foundation for such choices in the space of risky prospects.

[15] However, such a choice correspondence may not satisfy Sen's α. See, for instance, Case 2 of the example in Section I.B of Manzini and Mariotti (2007).

made under a complete but not necessarily transitive binary relation $\succsim$, including papers on voting (Miller, 1977, for example). Such a binary relation is often referred to as a nontransitive preference.[16] In the context of social choices, a nontransitive preference may emerge when aggregating the preferences of multiple individuals. In the context of individual decision-making, a nontransitive preference may be obtained from the DM's choices over binary menus. As we have noted, if the asymmetric part of the nontransitive preference is not acyclic, then using the set of undominated alternatives $\mathcal{U}_{\succsim}(A)$ to describe the choices made in a menu A may result in an empty set of choices.

The literature offers remedies for the above problem by suggesting that the DM's choices are those undominated with respect to the transitive closure of the nontransitive preference $\succsim$ in each menu (Kalai, Pazner and Schmeidler, 1976; Kalai and Schmeidler, 1977).[17] Specifically, for a given menu A, the transitive closure of $\succsim$ in A, denoted by $\succsim_A^+$, is defined such that for all $x, y \in A$, $x \succsim_A^+ y$ if and only if there is a sequence of finite alternatives $(x_k)_{k=1}^n$ in A such that $x_1 = x$, $x_n = y$, and for $k = 1, ..., n-1$, $x_k \succsim x_{k+1}$. The DM chooses $f(A) = \mathcal{U}_{\succsim_A^+}(A)$ in each menu A. Choices in $\mathcal{U}_{\succsim_A^+}(A)$ are called the *top-cycle choices* in A. As shown by Theorem 5.1 of Evren, Nishimura and Ok (2019), the choice correspondence that selects top-cycle choices satisfies the Weak Arrow's Choice axiom, which is a stronger version of Sen's β.[18] Therefore, the model of top-cycle choices intersects with our richness condition at rationality.

Covering relation. The covering relation, proposed and studied by Fishburn (1977) and Miller (1980), is an important welfare principle in the literature on nontransitive preferences. For a given nontransitive preference $\succsim$ and a menu A, the covering relation $\succsim_A^{cov}$ over A is defined such that for all $x, y \in A$, $x \succsim_A^{cov} y$ if for all $z \in A$, $y \succsim z$ implies $x \succsim z$. Clearly, $\succsim_A^{cov}$ is transitive, and thus the set $\mathcal{U}_{\succsim_A^{cov}}(A)$ is nonempty. Choices in $\mathcal{U}_{\succsim_A^{cov}}(A)$ can be considered as those that are efficient under the welfare principle. If for every menu A, the DM's set of choices $f(A)$ coincides with $\mathcal{U}_{\succsim_A^{cov}}(A)$, then f satisfies Condorcet Consistency. It follows that this class of choice models are equivalent to rationality under Comparative Richness.

To see why Condorcet Consistency is satisfied, note that $x \succsim y$, $x \succsim_{\{x,y\}}^{cov} y$,

[16] By our terminology, a preference is a nontransitive preference that satisfies transitivity.

[17] See also Ehlers and Sprumont (2008) and Evren, Nishimura and Ok (2019) for more discussions and characterizations of this choice model.

[18] The Weak Arrow's Choice axiom states that for all $x \in X$ and $A, B \in \mathcal{M}(X)$ with $x \in B \subseteq A$, if $x \in f(A)$, then $f(B) \subseteq f(A)$.

and $x \in f(\{x, y\})$ are all equivalent. On the one hand, if $\mathcal{W}(A)$ is not empty, then for every Condorcet winner x in A, we have that for every $z \in A$, $x \succsim z$. It follows that for every $y \in A$, $x \succsim_A^{cov} y$. Thus, each Condorcet winner in A must be chosen. On the other hand, consider any two alternatives $x \in \mathcal{W}(A)$ and $y \in A \backslash \mathcal{W}(A)$. There is $z \in A$ such that $x \succsim z$ but $z \succ y$, which implies $y \not\succsim_A^{cov} x$. It follows that $x \succ_A^{cov} y$ and $y \notin \mathcal{U}_{\succsim_A^{cov}}(A)$. Thus, only alternatives in $\mathcal{W}(A)$ are chosen.

4.2 Revisiting Other Classic Axioms

Based on the preceding discussions, under Comparative Richness, many classic axioms are sufficient for rationality. In this section, we study axioms that lead to nontrivial bounded rational choice models under Comparative Richness.

4.2.1 Aizerman's Axiom and Reducibility

We begin with Aizerman's axiom.

Axiom 6 (Aizerman)**.** *For all $A \in \mathcal{M}(X)$ and $x \in A \backslash f(A)$, $f(A) = f(A \backslash \{x\})$.*

Aizerman's axiom states that deleting unchosen alternatives in a given menu does not affect the set of chosen alternatives. The axiom was first introduced by Chernoff (1954) and referred to as the property of independence of rejecting outcast variants by Aizerman and Malishevski (1981) and Aizerman (1985).

Aizerman's axiom is satisfied by many choice models. For instance, the choice correspondence that selects undominated alternatives under a transitive binary relation satisfies this axiom.[19] Other choice models that satisfy the axiom include the top-cycle choice model we discussed in Section 4.1 and the model of choice with a preference structure introduced by Evren, Nishimura and Ok (2019).[20] Aizerman and Malishevski (1981) and Moulin (1985) show that

[19]To see this, consider a transitive binary relation $\succsim$. Note that its asymmetric part $\succ$ is also transitive. Thus, for any menu A, each alternative in $A \backslash \mathcal{U}_{\succsim}(A)$ must be $\succ$-dominated by some alternative in $\mathcal{U}_{\succsim}(A)$. Hence, for every $x \in A \backslash \mathcal{U}_{\succsim}(A)$, we have $\mathcal{U}_{\succsim}(A) = \mathcal{U}_{\succsim}(A \backslash \{x\})$.

[20]A preference structure is a tuple $(\succsim, \succsim^*)$ such that $\succsim$ is a complete binary relation (nontransitive preference) over X, and $\succsim^*$ is a reflexive and transitive binary relation over X, with certain consistency conditions imposed on the tuple (Nishimura and Ok, 2018). Evren, Nishimura and Ok (2019) study choices with a preference structure $(\succsim, \succsim^*)$ whereby for each menu A, the DM considers the set of top-cycle choices B in A under $\succsim$, and applies $\succsim^*$ to eliminate strictly dominated choices. Theorem 5.2 of Evren, Nishimura and Ok (2019) demonstrates that the model of choice with a preference structure satisfies Aizerman's axiom. We note that the top-cycle choice model is a special case of the model of choice with a preference structure when the second binary relation $\succsim^*$ is the trivial one—i.e., for all $x, y \in X$, $x \succsim^* y$ if and only if $x = y$.

Aizerman's axiom and Sen's α can jointly characterize choice correspondences that are rationalizable by multiple preferences. That is, a choice correspondence f satisfies the two axioms if and only if there exists a nonempty set of preferences $\{\succsim_k\}_{k\in I}$ over X such that for every menu A, $f(A) = \bigcup_{k\in I} \mathcal{U}_{\succsim_k}(A)$.

We introduce a weaker version of Aizerman's axiom, which is studied by Li, Tang and Zhang (2023) to characterize how DMs associate alternatives with each other in her consideration set.

Axiom 7 (Reducibility)**.** *For all $A \in \mathcal{M}(X)$, if for all $x \in A$, $f(A\backslash\{x\}) \neq f(A)$, then $f(A) = A$.*

We say that a menu A is *invariant* if $f(A) = A$. We interpret Reducibility by considering its contrapositive: For every menu A, if A is not invariant, then there exists $x \in A$ such that $f(A\backslash\{x\}) = f(A)$. Note that such an alternative x cannot be chosen in $f(A)$. Thus, the axiom states that we can find one alternative that is not chosen, such that deleting it has no impact on the set of chosen ones. This axiom is satisfied, for instance, by the choice correspondence that selects undominated alternatives with respect to an acyclic binary relation, while Aizerman's axiom may not.[21] Nevertheless, the following theorem asserts that under Comparative Richness, Aizerman's axiom and Reducibility lead to the same choice model.

Theorem 4. *If (X, f) satisfies Comparative Richness, then the following statements are equivalent:*

(1) *(X, f) satisfies Aizerman's axiom;*

(2) *(X, f) satisfies Reducibility;*

(3) *For every menu A, $f(A)$ is invariant and $A \bowtie^f f(A)$;*

(4) *There exists a linear order $\trianglerighteq$ over $\mathcal{M}(X)$ such that for all menus A and B with $B \subsetneq A$, $f(A) \trianglerighteq B$.*

We will refer to the linear order $\trianglerighteq$ in (4) as the *implied menu preference*, and f is said to be *MP-rationalized* by $\trianglerighteq$.[22] Essentially, the DM's choices can be

[21]Consider an acyclic binary relation $\succ$ over X. Note that an acyclic binary relation must be asymmetric, and thus the asymmetric part of $\succ$ is just $\succ$ itself. For any menu A, if $A\backslash\mathcal{U}_{\succ}(A)$ is not empty, by acyclicity we can find some $y \in A\backslash\mathcal{U}_{\succ}(A)$ such that y does not dominate any other alternative in $A\backslash\mathcal{U}_{\succ}(A)$ with respect to $\succ$. It follows that y does not dominate any other alternative in A. Deleting such an alternative y will not affect the set of alternatives that are $\succ$-undominated in A. Thus, $\mathcal{U}_{\succ}(A) = \mathcal{U}_{\succ}(A\backslash\{y\})$. Therefore, Reducibility is satisfied. To see that Aizerman's axiom may be violated, consider $X = \{x, y, z\}$ and a binary relation $\succ = \{(x, y), (y, z)\}$. We have $\mathcal{U}_{\succ}(X) = \{x\}$, while $\mathcal{U}_{\succ}(X\backslash\{y\}) = \{x, z\}$.

[22]Recall that "MP" stands for "menu preference."

interpreted as committing to her favorite submenu according to $\trianglerighteq$, anticipating that each choice in the submenu will be optimal under some contingency. The fact that the DM does not want to commit to a single alternative may be due to a preference for flexibility (Kreps, 1979) or the anticipation of information before the eventual choice (Dillenberger, Lleras, Sadowski and Takeoka, 2014).

To see that statement (3) implies (4), we construct the linear order $\trianglerighteq$ in (4) such that every invariant menu is ranked strictly higher than every non-invariant menu. Among invariant menus, we only need to guarantee that $\trianglerighteq$ is consistent with the shadow preference $\trianglerighteq^f$ with ties being broken by the set inclusion order—i.e., for any two distinct invariant menus A and B, $A \triangleright B$ if either (i) $A \triangleright^f B$ or (ii) $A \bowtie^f B$ and $B \subseteq A$. Such a linear order $\trianglerighteq$ MP-rationalizes f. To see this, consider a menu A. Note that for any invariant $B \in \mathcal{M}(A)$, either $f(A) \bowtie^f A \triangleright^f B$ or $f(A) \bowtie^f A \bowtie^f B$. By monotonicity of $\trianglerighteq^f$, $f(A) \bowtie^f A \bowtie^f B$ implies $f(B) = B \subseteq f(A)$. Therefore, $B \neq f(A)$ implies $f(A) \triangleright B$.

We note that statement (4) implies (1) even without Comparative Richness. That is, any choice correspondence f that is MP-rationalized by some linear order $\trianglerighteq$ necessarily satisfies Aizerman's axiom. To see this, consider a menu A and an alternative $x \in A \backslash f(A)$. It follows that $f(A) \in \mathcal{M}(A \backslash \{x\})$ and for all $B \in \mathcal{M}(A \backslash \{x\})$, $f(A) \trianglerighteq B$. Therefore, we have $f(A \backslash \{x\}) = f(A)$.

In the Online Appendix, we provide two examples: One shows that MP-rationalizability and Comparative Richness together cannot imply rationalizability; the other shows that without Comparative Richness, Aizerman's axiom alone cannot imply MP-rationalizability.

4.2.2 Weaker Axiom of Revealed Preference

The next axiom we study is the Weaker Axiom of Revealed Preference, which is a weaker version of the Weak Axiom of Revealed Preference.

Axiom 8 (Weak Axiom of Revealed Preference (WARP))**.** *For all* $A, B \in \mathcal{M}(X)$ *and* $x, y \in A \cap B$, *if* $x \in f(A)$ *and* $y \notin f(A)$, *then* $y \notin f(B)$.[23]

Axiom 9 (Weaker Axiom of Revealed Preference (WrARP))**.** *For all* $A, B \in \mathcal{M}(X)$ *and* $x, y \in A \cap B$, *if* $x \in f(A)$, $y \notin f(A)$, *and* $x \notin f(B)$, *then* $y \notin f(B)$.

WARP entails a revealed preference interpretation: If an alternative x is chosen in A while y is not, then it is revealed that x is strictly better than y.

[23] The standard statement of WARP is as follows: For all $A, B \in \mathcal{M}(X)$ and $x, y \in A \cap B$, if $x \in f(A)$ and $y \in f(B)$, then $x \in f(B)$. This statement is equivalent to the one above.

Thus, whenever x is feasible in menu B, y cannot be chosen. Similarly, WrARP states that if x is revealed to be strictly better than y, then y cannot be revealed to be strictly better than x. It is well known that a choice correspondence satisfies WARP if and only if it is rationalizable, and WrARP is not sufficient for rationalizability. In the literature, WrARP appears as Axiom 3 in Jamison and Lau (1973) and is shown in Fishburn (1975) to be one of the key axioms that characterize the choice of undominated alternatives under an interval order.[24]

To proceed, we introduce the choice model that can be characterized by WrARP under Comparative Richness. For a given preference $\succsim$ over X, we say that y is *just better* than x if $y \succ x$ and there does not exist alternative $z \in X$ such that $y \succ z \succ x$. For each alternative x, denote by $J^+_{\succsim}(x)$ the set of all alternatives that are just better than x. Let $\mathcal{G}(X) \subseteq X \times \mathcal{M}(X)$ be such that $(x, A) \in \mathcal{G}(X)$ if and only if $x \in A$.

Definition 5. *Let $\succsim$ be a preference over X and $\gamma : \mathcal{G}(X) \to X$ be a function. The pair $(\succsim, \gamma)$ is a monotone overevaluation system if the following conditions are satisfied for all $A, B \in \mathcal{M}(X)$ and $x, y \in A \cap B$:*

(1) $\gamma(x, A) \in \{x\} \cup J^+_{\succsim}(x)$;

(2) *If $B \subseteq A$ and $\gamma(x, B) \in J^+_{\succsim}(x)$, then $\gamma(x, A) \in J^+_{\succsim}(x)$;*

(3) *If $x \sim y$, $\gamma(x, A) = x$, $\gamma(y, A) \in J^+_{\succsim}(y)$, and $\gamma(x, B) \in J^+_{\succsim}(x)$, then $\gamma(y, B) \in J^+_{\succsim}(y)$.*

A choice correspondence (X, f) is MO-rationalized[25] *by the monotone overevaluation system $(\succsim, \gamma)$ if for all $A \in \mathcal{M}(X)$, $x \in f(A)$ if and only if for all $y \in A$, $\gamma(x, A) \succsim \gamma(y, A)$. A choice correspondence (X, f) is MO-rationalizable if there is a monotone overevaluation system that MO-rationalizes (X, f).*

By Definition 5, for a given monotone overevaluation system $(\succsim, \gamma)$, each alternative $x \in A$ is evaluated according to alternative $\gamma(x, A)$. There are two cases: The DM can either evaluate x correctly (i.e., $\gamma(x, A) = x$) or make a mistake by slightly overevaluating x (i.e., $\gamma(x, A) \in J^+_{\succsim}(x)$), possibly due to ignorance of some inferior attribute of x. Note that $\gamma(x, A) \in \{x\} \cup J^+_{\succsim}(x)$ means that the DM only makes minor mistakes.

The notable conditions in Definition 5 are (2) and (3). Condition (2) states that the DM is more likely to make mistakes when she faces a larger menu.

[24] An interval order is an asymmetric binary relation $\succ$ over X such that for all $x, y, z, w \in X$, if $x \succ y$ and $z \succ w$, then either $x \succ w$ or $z \succ y$. See Fishburn (1970) for more details.

[25] Recall that "MO" stands for "monotone overevaluation."

Condition (3) states that equally desirable alternatives can be ranked according to the DM's tendency to overevaluate them.

Theorem 5. *If (X, f) satisfies Comparative Richness, then the following statements are equivalent:*

(1) *(X, f) satisfies WrARP;*

(2) *(X, f) is MO-rationalizable;*

(3) *(X, f) is MO-rationalized by a monotone overevaluation system $(\succsim, \gamma)$ such that for all $x, y \in X$, $x \succsim y$ if and only if $\{x\} \trianglerighteq^f \{y\}$.*

In the construction of the monotone overevaluation system, the key step is to show that for all menu A and $x \in f(A)$, there does not exist any $y \in X$ such that $A \rhd^f \{y\} \rhd^f \{x\}$. This property of the shadow preference implies that there can only be a slight overevaluation; that is, each alternative can at most be mistakenly evaluated as some alternative that is just better.

Again, even without Comparative Richness, the fact that f is MO-rationalized by a monotone overevaluation system $(\succsim, \gamma)$ is sufficient for it to satisfy WrARP. To see this, suppose to the contrary that we can find two menus A and B and two alternatives x and y contained in both menus such that $x \in f(A)$, $y \notin f(A)$, $x \notin f(B)$, but $y \in f(B)$. It follows that $\gamma(x, A) \succ \gamma(y, A)$ and $\gamma(y, B) \succ \gamma(x, B)$. It is clear that we must have $x \sim y$. Thus, x is overevaluated in A but not in B, and y is overevaluated in B but not in A, which contradicts condition (3) of Definition 5.

In the Online Appendix, we provide two examples related to Theorem 5: One example shows that WrARP and Comparative Richness together cannot imply rationalizability and the other shows that without Comparative Richness, WrARP alone is not sufficient for MO-rationalizability.

4.2.3 WARP with Limited Attention

In this section, we study choices with limited attention introduced by Masatlioglu, Nakajima and Ozbay (2012) (henceforth MNO) under Comparative Richness. The following axiom is a natural generalization of the Weak Axiom of Revealed Preference with Limited Attention (WARP(LA)) proposed by MNO.

Axiom 10 (WARP(LA)*)**.** *For every $A \in \mathcal{M}(X)$, there exists $x \in A$ such that for every $B \in \mathcal{M}(X)$, if $f(B) \cap A \neq \emptyset$ and $f(B) \neq f(B \backslash \{x\})$, then $x \in f(B)$.*

WARP(LA)* states that for every given menu A, there is an alternative x such that if x is considered in some menu B (which is revealed from the fact that deleting x from B affects the set of chosen alternatives), and an alternative in A is chosen in B, then x must be chosen in B. Intuitively, if x is the best alternative in A and is considered in B, then if any alternative in A is chosen in B, the weakly better alternative x should also be chosen in B.

In contrast to our paper, MNO focus on choice functions: A choice correspondence f is said to be a *choice function* if for every menu A, $|f(A)| = 1$.[26] If f is a choice function, WARP(LA)* reduces to the requirement that for every menu A, there exists $x \in A$ such that for every menu B, if $f(B) \in A$ and $f(B) \neq f(B\backslash\{x\})$, then $f(B) = x$. This statement is precisely MNO's WARP(LA). In this sense, WARP(LA)* can be viewed as a natural generalization of WARP(LA) to choice correspondences.

MNO show that WARP(LA) characterizes a DM who pays attention to a subset of the feasible alternatives and chooses the best alternative among them. Formally, a function $\Gamma : \mathcal{M}(X) \to \mathcal{M}(X)$ is said to be an *attention filter* if for every menu A, (i) $\Gamma(A) \subseteq A$ and (ii) for every $x \in A\backslash\Gamma(A)$, $\Gamma(A\backslash\{x\}) = \Gamma(A)$.[27] To interpret, $\Gamma(A)$ is the subset of alternatives the DM pays attention to in menu A. This set is unaffected by the removal of alternatives in A to which the DM does not pay attention.

Definition 6. *A choice correspondence (X, f) is LA-rationalizable*[28] *if there is a tuple $(\succsim, \Gamma)$, where $\succsim$ is a preference over X and Γ is an attention filter, such that for every menu A, $f(A) = \mathcal{U}_{\succsim}(\Gamma(A))$. The tuple $(\succsim, \Gamma)$ is said to LA-rationalize f.*

Whereas MNO show that a choice function satisfies WARP(LA) if and only if it is LA-rationalizable, for general choice correspondences, WARP(LA)* may not guarantee LA-rationalizability.

Definition 7. *A choice correspondence (X, f) is GA-rationalizable*[29] *if there exists a preference $\succsim$ over X, an attention filter Γ, and a choice function τ such that for every menu A, $f(A) = \{x \in \Gamma(A) : x \succsim \tau(\Gamma(A))\}$. The tuple $(\succsim, \Gamma, \tau)$ is said to GA-rationalize (X, f).*

[26] When f is a choice function, we will abuse the notation a little bit by treating $f(A)$ as an alternative instead of a singleton menu.
[27] Basically, an attention filter is a choice correspondence that satisfies Aizerman's axiom.
[28] "LA" stands for "limited attention."
[29] "GA" stands for "generalized choices with limited attention."

GA-rationalizability posits that for any given menu A, the DM considers $\Gamma(A)$ and accepts each alternative in $\Gamma(A)$ that is weakly better than a benchmark alternative $\tau(\Gamma(A))$. Note that if f is a choice function and can be GA-rationalized by $(\succsim, \Gamma, \tau)$, then it is necessary that τ selects the $\succsim$-best alternative in $\Gamma(A)$. Thus, for choice functions, GA-rationalizability is equivalent to LA-rationalizability.

Theorem 6. *A choice correspondence (X, f) is GA-rationalizable if and only if it satisfies WARP(LA)*.*

We proceed to study the joint implication of WARP(LA)* and Comparative Richness. To this end, we will show that Comparative Richness leads to a particular class of attention filters—namely, ordered attention filters.

Formally, a function $\Gamma : \mathcal{M}(X) \to \mathcal{M}(X)$ is said to be an *ordered attention filter* if there is a linear order $\trianglerighteq$ over $\mathcal{M}(X)$ such that for every menu A, $\Gamma(A)$ is the unique element in $\mathcal{U}_{\trianglerighteq}(\mathcal{M}(A))$.[30] We will use $\Gamma_{\trianglerighteq}$ as shorthand for the ordered attention filter induced by the linear order $\trianglerighteq$. One possible interpretation is that $\trianglerighteq$ captures the salience of bundles of alternatives, and in every menu A, the DM pays attention to the most salient bundle of alternatives in A.

Definition 8. *A choice correspondence (X, f) is OA-rationalizable*[31] *if there exists a preference $\succsim$ over X, an ordered attention filter Γ, and a choice function τ such that for every menu A, $f(A) = \{x \in \Gamma(A) : x \succsim \tau(\Gamma(A))\}$. The tuple $(\succsim, \Gamma, \tau)$ is said to OA-rationalize (X, f).*

The following theorem demonstrates that under Comparative Richness, WARP(LA)*, GA-rationalizability, and OA-rationalizability are all equivalent.

Theorem 7. *If (X, f) satisfies Comparative Richness, then the following statements are equivalent:*

(1) *(X, f) satisfies WARP(LA)*;*

(2) *(X, f) is GA-rationalizable;*

(3) *(X, f) is OA-rationalizable.*

We briefly demonstrate why GA-rationalizability implies OA-rationalizability under Comparative Richness. For a given choice correspondence (X, f) that

[30] Basically, an ordered attention filter is an MP-rationalizable choice correspondence. Recall that MP-rationalizable choice correspondences necessarily satisfy Aizerman's axiom. Thus, any ordered attention filter is an attention filter.

[31] "OA" stands for "ordered attention."

is GA-rationalizable, we can construct a tuple $(\succsim, \Gamma, \tau)$ that GA-rationalizes f and satisfies that for every menu A, $\Gamma(A) \bowtie^f A$. Following a procedure similar to the proof of Theorem 4, we can start with the shadow preference $\trianglerighteq^f$, break ties properly, and end up with a linear order $\trianglerighteq$ such that for all menu A and submenu $B \subseteq A$, $\Gamma(A) \trianglerighteq B$. Thus, the tuple $(\succsim, \Gamma, \tau)$ also OA-rationalizes f.

In the Online Appendix, we present an example that demonstrates that without Comparative Richness, GA-rationalizability is insufficient for OA-rationalizability.

We conclude this section by noting that the OA-rationalizability of a given choice function (X, f) reduces to the existence of a tuple $(\succsim, \Gamma)$, where $\succsim$ is a linear order over X and Γ is an ordered attention filter, such that for every menu A, $f(A) \succsim y$ for every $y \in \Gamma(A)$. The OA-rationalizability of choice functions is referred to as the *choose twice* procedure and has been shown to be *equivalent* to the LA-rationalizability in Richter (2020). Since for choice functions LA-rationalizability is equivalent to GA-rationalizability, Theorem 6 and the related results in Richter (2020) jointly demonstrate the equivalence between GA-rationalizability and OA-rationalizability when the choice correspondence is either a choice function or satisfies comparative richness.

5 Quantifying Departures from Rationality

In this section, we propose new axioms that explicitly quantify the departure from rationality. We then demonstrate that these axioms lead to novel choice models under Comparative Richness. Throughout the section, assume $T \in \mathbb{N}_+$.

Axiom 11 (Weak Sen's α). *For all $A, B \in \mathcal{M}(X)$ and distinct $x, y \in X$ with $x, y \in B \subseteq A$, if $x \in f(A)$ and $x \notin f(A \backslash \{y\})$, then $x \in f(B)$.*

Axiom 12 (T-Weak Sen's α). *For all $A, B, C \in \mathcal{M}(X)$ and $x \in X$ with $x \in B \backslash C$, $C \subsetneq B \subseteq A$, and $|C| = T$, if $x \in f(A)$ and for every $y \in C$, $x \notin f(A \backslash \{y\})$, then $x \in f(B)$.*

Weak Sen's α is equivalent to T-Weak Sen's α when $T = 1$, and for all $T \in \mathbb{N}_+$, T-Weak Sen's α implies $(T+1)$-Weak Sen's α. Weak Sen's α states that if a violation of Sen's α occurs on the choice of alternative x when y is removed from the menu, then no other alternatives can trigger another violation of Sen's α on the choice of x. Thus, when both x and y are preserved in the menu, Sen's α holds for x. Similarly, the T-Weak Sen's α axiom allows for at most T distinct alternatives that can trigger violations of Sen's α on the choice of any given alternative x.

To present the representations characterized by the axioms above, we will need several definitions. For any $T \in \mathbb{N}_+$, we use $\succsim^T$ to denote a generic preference over X^T. For any subset $F \subseteq X^T$, let $F|_1 = \{x_1 \in X : (x_1, ..., x_T) \in F\}$ be the projection of F on its first dimension.

Definition 9. *For any $T \geq 2$, a choice correspondence (X, f) is T-rationalizable if there is a preference $\succsim^T$ over X^T such that for all $A \in \mathcal{M}(X)$, $f(A) = \mathcal{U}_{\succsim^T}(A^T)|_1$. Such $\succsim^T$ is said to T-rationalize (X, f).*

T-rationalizability captures the situation in which the DM's evaluation of each alternative depends on the presence of other feasible alternatives in the menu. In particular, $(x_1, ..., x_T) \succsim^T (y_1, ..., y_T)$ means that the desirability of x_1, given the feasibility of $\{x_t\}_{t=2}^T$, is weakly higher than the desirability of y_1, given the feasibility of $\{y_t\}_{t=2}^T$. The parameter T in Definition 9 represents the maximum number of alternatives that can simultaneously influence the DM's evaluation of a given alternative. The alternatives chosen by the DM are those that maximize this preference.

Theorem 8. *If (X, f) satisfy Comparative Richness, then the following statements are equivalent for all $T \in \mathbb{N}_+$:*

(1) *(X, f) satisfies T-Weak Sen's α;*

(2) *(X, f) is $(T+1)$-rationalizable;*

(3) *For all menu A and $x \in f(A)$, there is a menu $B \subseteq A$ with $|B| \leq T+1$ such that $A \bowtie^f B$ and $x \in f(B)$.*

We have several remarks on Theorem 8. First, without Comparative Richness, $(T+1)$-rationalizability alone is sufficient for T-Weak Sen's α. Second, without Comparative Richness, T-Weak Sen's α alone does not imply $(T+1)$-rationalizability. We provide an example in the Online Appendix.

We note that the "choice with attraction effects" example in the Introduction and Section 3.2 satisfies Weak Sen's α and Comparative Richness. Clearly, it is 2-rationalizable by the utility function $v^* : \mathbb{R}^{2n} \to \mathbb{R}$, which satisfies that for all $x, y \in \mathbb{R}^n$, $v^*(x, y) = v^+(x)$ if x dominates y, and $v^*(x, y) = v(x)$ otherwise.

In fact, 2-rationalizability can be naturally applied to choice over risk and uncertainty. In many theories on nontransitive preferences over lotteries, including the salience theory (Bordalo, Gennaioli and Shleifer, 2012) and regret theory (Bell, 1982; Loomes and Sugden, 1982; Fishburn, 1989), a nontransitive preference $\succsim$ over the space of lotteries X is represented by some function $v : X^2 \to \mathbb{R}$ such that $x \succsim y$ if and only if $v(x, y) \geq v(y, x)$. This can be viewed as a special case of the 2-rationalizability restricted on binary menus. To see this, consider some function $u : X^2 \to \mathbb{R} \cup \{-\infty\}$ such that for all $x, y \in X$, $u(x, x) = -\infty$ and $u(x, y) = v(x, y)$ if $x \neq y$. The choice correspondence that is 2-rationalized by u induces the same choices in binary menus as v. In this regard, our notion of 2-rationalizability provides a straightforward extension of these nontransitive preferences to choice correspondences over general menus. By defining $v(x', x') = -\infty$ for every $x' \in X$, in each menu A, an alternative x is chosen in A if and only if for every alternative $y \in A$, $\max_{z \in A} v(x, z) \geq \max_{w \in A} v(y, w)$.

Conversely, we can start with a choice correspondence that is 2-rationalized by some u, and study its implications on binary menus by considering the nontransitive preferences $v(x, y) = \max\{u(x, x), u(x, y)\}$. The following example demonstrates that a quadratic form for u captures the idea of disappointment aversion. The nontransitive preference induced by u intuitively exhibits both Allais-type behavior and preference reversal. Our example highlights a new approach to modeling nontransitive preferences over lotteries.

Choice with bounded disappointment aversion. Let X be the set of simple lotteries over $\mathbb{R}$—i.e., each lottery $x \in X$ takes the form of $x = (x_1, p_1; ...; x_n, p_n)$ with the interpretation that x yields monetary payoff x_k with probability p_k for each $k \in \{1, ..., n\}$. Consider the utility function $u : X^2 \to \mathbb{R}$ such that for any $x, y \in X$ with $x = (x_1, p_1; ...; x_n, p_n)$ and $y = (y_1, q_1; ...; y_m, q_m)$, we have

$$u(x, y) = \sum_{k=1}^{n} p_k v(x_k) - \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j \lambda \min\{\max\{v(y_j) - v(x_i), 0\}, L\}, \tag{1}$$

where $L > 0$, $\lambda \in (0, 1)$, and $v : \mathbb{R} \to \mathbb{R}$ is strictly increasing and satisfies $v(\mathbb{R}) = \mathbb{R}$.[32] The term $V(x) := \sum_{k=1}^{n} p_k v(x_k)$ is the DM's normative utility, and the second term $D(x, y) := \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j \lambda \min\{\max\{v(y_j) - v(x_i), 0\}, L\}$ captures the DM's disappointment aversion, where L is the maximal disappointment that the DM can perceive. The disappointment comes from the risk of the lottery, and thus the DM can perceive disappointment even when there is only one feasible choice—i.e., $D(x, x) > 0$ if x is not a degenerate lottery. Let f be the

[32] Our functional form is a special case of the general quadratic form introduced in footnote 45 of Machina (1982).

choice correspondence that is 2-rationalized by u.

We show that f satisfies Comparative Richness. For any given lottery x, the lottery $y = (y_1, 1/2; y_2, 1/2)$ is independent of x as long as $v(y_1)$ is sufficiently large and $v(y_2)$ is sufficiently small. This is because in this case, we have $D(x, x) \leq \lambda L/4 < \lambda L/2 = D(x, y)$ and $D(y, y) = \lambda L/4 < \lambda L/2 = D(y, x)$, which indicates that y plays no role in evaluating x, and vice versa. To show Comparative Richness, consider any menu A and a submenu B contained in A. The alternative $y = (y_1, 1/2; y_2, 1/2)$ that satisfies $A \perp \{y\}$ and $u(y, y) = \max_{x,z\in B} u(x, z)$ meets the requirement of Comparative Richness.

Our first observation of f is that it accommodates the Allais paradox (Allais, 1953). Specifically, consider lotteries $x = (4000, 4/5; 0, 1/5)$, $y = (3000, 1)$, $z = (4000, 1/5; 0, 4/5)$, and $w = (3000, 1/4; 0, 3/4)$. Observe that z and w are mixtures of x and y with lottery $(0, 1)$ under the same mixture ratio 1/4, respectively. While the Independence Axiom asserts that x is chosen over y if and only if z is chosen over w, the Allais paradox intuitively suggests that even for DMs who choose z in $\{z, w\}$, they still tend to choose y in $\{x, y\}$, since y is a certain outcome.[33] We note that our choice correspondence f, like many other theories of nontransitive preferences, accommodates the Allais paradox. For instance, when $v(0) = 0$, $v(3000) = 100$, $v(4000) = 130$, $\lambda = 1/2$, and $L = 500$, we have $f(\{x, y\}) = \{y\}$ and $f(\{z, w\}) = \{z\}$.

Our second observation is that f accommodates the phenomenon of preference reversal (Lichtenstein and Slovic, 1971; Lindman, 1971; Grether and Plott, 1979). The phenomenon refers to the situation in which for two lotteries the DM attaches the same certainty equivalent to, she prefers to choose the low-risk lottery over the high-risk one. Formally, consider two lotteries $x = (a, p; 0, 1-p)$ and $y = (b, q; 0, 1-q)$ such that $a > b > 0$ and $q > p$. Lottery x is the high-risk lottery and y is the low-risk one. Suppose that for some lottery $z = (c, 1)$, we have $f(\{x, z\}) = \{x, z\}$ and $f(\{y, z\}) = \{y, z\}$. Then the phenomenon of preference reversal refers to the choice in which $f(\{x, y\}) = \{y\}$.

In the Online Appendix, we show that the preference reversal always happens for f when L is relatively large. Here, we give a numerical example for demonstration. For simplicity, assume that for all $a \in \mathbb{R}$, $v(a) = a$. Let $x = (30, 1/2; 0, 1/2)$, $y = (21, 2/3; 0, 1/3)$, $z = (12, 1)$, $\lambda = 1/2$, and $L = 500$. We have $u(z, z) = u(x, z) = 12 > u(x, x) = 45/4 > u(z, x) = 15/2$ and $u(z, z) = u(y, z) = 12 > u(y, y) = 35/3 > u(z, y) = 9$. Note that when the

[33] Evidence of the Allais paradox can be found in Kahneman and Tversky (1979).

DM is indifferent between a risky lottery and a certain payoff, she takes the certain payoff as the reference point because she can minimize her perceived disappointment by doing so. For the comparison between the two risky lotteries, we have $u(y, y) > u(x, y) = 23/2 > u(x, x) > u(y, x) = 10$. Hence, the DM minimizes her perceived disappointment by taking the low-risk lottery as the reference point. Note that the two risky lotteries generate the same u when the certain payoff is taken as the reference point, and the high-risk lottery's u is decreased more after the reference point shifts from the certain payoff to the low-risk lottery. Thus, the low-risk lottery is chosen in the binary menu $\{x, y\}$.

We end this section by noting that Sen's β can be weakened in a similar fashion. In the Online Appendix, we formally define T-Weak Sen's β and demonstrate that even with Comparative Richness, T-Weak Sen's α is not equivalent to T-Weak Sen's β. We leave the characterization of T-Weak Sen's β to future research.

Appendix

Proof of Theorem 1. First, we prove statements (1) and (2). Assume $f(A) \cap A_i \neq \emptyset$, and consider an arbitrary $x \in A \backslash A_i$. For each $y \in f(A) \cap A_i$, $x \perp y$ implies $y \in f(A \backslash \{x\}) \cap A_i$. Thus, $f(A) \cap A_i \subseteq f(A \backslash \{x\}) \cap A_i$. By induction, we have $f(B) \cap A_i \neq \emptyset$. Furthermore, pick some $y \in f(A) \cap A_i$, and $x \perp y$ implies $f(A \backslash \{x\}) \subseteq f(A)$. Thus, $f(A) \cap A_i = f(A \backslash \{x\}) \cap A_i$. By induction, we have $f(A) \cap A_i = f(A_i)$. To see statement (3), note that since $\{A_k\}_{k \in I}$ is an MIC, $\{A_k\}_{k \in I \backslash \{i,j\}} \cup \{A_i \cup A_j\}$ is also an MIC. By statement (1), $f(A) \cap (A_i \cup A_j) \neq \emptyset$ and $f(B) \cap (A_i \cup A_j) \neq \emptyset$ implies $f(A) \cap (A_i \cup A_j) = f(B) \cap (A_i \cup A_j) = f(A_i \cup A_j)$. Therefore, $f(B) \cap A_j \neq \emptyset$ implies $f(A) \cap A_j \neq \emptyset$. □

Proof of Theorem 2. Let $\trianglerighteq^f$ be the shadow order of choice correspondence (X, f). If $\trianglerighteq^f$ is a preference, then for any two menus A and B with $B \subseteq A$, we can find $x \perp A$ and $y \perp A$ such that $A \approx^f x$ and $B \approx^f y$. The existence of such an alternative y implies Comparative Richness. For the rest of the proof, we assume that Comparative Richness holds.

First, we show statement (1). Consider two menus A and B. We want to show that for all $x, y \in X$ with $x \approx^f A$ and $y \approx^f B$ such that $\{\{x\}, \{y\}, A \cup B\}$ is an MIC, we have that $A \trianglerighteq^f B$ implies $x \in f(\{x, y\})$ (the inverse is true by the definition of $\trianglerighteq^f$). To see this, consider any such alternatives x and y and note that by $A \trianglerighteq^f B$, there exist $\hat{x}, \hat{y} \in X$ such that $\{\{\hat{x}\}, \{\hat{y}\}, A \cup B\}$ is an MIC with $\hat{x} \approx^f A$, $\hat{y} \approx^f B$, and $\hat{x} \in f(\{\hat{x}, \hat{y}\})$. By Comparative Richness, we can find $z, w \in X$ such that $\{\{x\}, \{y\}, \{z\}, \{w\}, A \cup B\}$ and $\{\{\hat{x}\}, \{\hat{y}\}, \{z\}, \{w\}, A \cup B\}$ are both MICs with $z \approx^f A$ and $w \approx^f B$. Consider the following MICs:

$$\Pi_A = \{\{x\}, \{y\}, \{z\}, \{w\}, A\},\ \Pi_B = \{\{x\}, \{y\}, \{z\}, \{w\}, B\},$$

$$\Omega_A = \{\{\hat{x}\}, \{\hat{y}\}, \{z\}, \{w\}, A\},\ \Omega_B = \{\{\hat{x}\}, \{\hat{y}\}, \{z\}, \{w\}, B\}.$$

Let $\trianglerighteq_{\Pi_A}, \trianglerighteq_{\Pi_B}, \trianglerighteq_{\Omega_A}$, and $\trianglerighteq_{\Omega_B}$ be the preferences that rationalize f on the four MICs, respectively. Since $x \approx^f A$ and $x \perp A$, by Corollary 1, we have $A \bowtie_{\Pi_A} \{x\}$. Similarly, we have

$$A \bowtie_{\Pi_A} \{z\},\ A \bowtie_{\Omega_A} \{\hat{x}\},\ A \bowtie_{\Omega_A} \{z\}, B \bowtie_{\Pi_B} \{y\},\ B \bowtie_{\Pi_B} \{w\},$$

$$B \bowtie_{\Omega_B} \{\hat{y}\},\ B \bowtie_{\Omega_B} \{w\},\ \{\hat{x}\} \trianglerighteq_{\Omega_A} \{\hat{y}\},\ \text{and } \{\hat{x}\} \trianglerighteq_{\Omega_B} \{\hat{y}\}.$$

By transitivity of those preferences, we have $\{x\} \bowtie_{\Pi_A} \{z\}$, $\{z\} \trianglerighteq_{\Omega_A} \{\hat{y}\}$, $\{\hat{y}\} \bowtie_{\Omega_B} \{w\}$, and $\{w\} \bowtie_{\Pi_B} \{y\}$. By Corollary 1, we have $\{x\} \bowtie_{\Pi_A} \{z\}$, $\{z\} \trianglerighteq_{\Omega_A} \{w\}$ ($\{z\} \trianglerighteq_{\Pi_A} \{w\}$), and $\{w\} \bowtie_{\Pi_A} \{y\}$. Thus, $\{x\} \trianglerighteq_{\Pi_A} \{y\}$, i.e., $x \in f(\{x, y\})$.

Next, we show that $\trianglerighteq^f$ is a preference. The completeness of $\trianglerighteq^f$ is guaranteed by Comparative Richness. To see the transitivity of $\trianglerighteq^f$, consider three menus A, B, and D such that $A \trianglerighteq^f B$ and $B \trianglerighteq^f D$. We show $A \trianglerighteq^f D$. By Comparative Richness, we can find $x, y, z \in X$ such that $\Pi = \{\{x\}, \{y\}, \{z\}, A \cup B \cup D\}$ is an MIC, $x \approx^f A$, $y \approx^f B$, and $z \approx^f D$. Let $\trianglerighteq_\Pi$ rationalize f on Π. Since $A \trianglerighteq^f B$ and $B \trianglerighteq^f D$, we have $\{x\} \trianglerighteq_\Pi \{y\}$ and $\{y\} \trianglerighteq_\Pi \{z\}$. Thus $\{x\} \trianglerighteq_\Pi \{z\}$, which implies $x \in f(\{x, z\})$. By the above argument, we have $A \trianglerighteq^f D$.

To see statement (2), consider an MIC $\Pi = \{A_k\}_{k \in I}$. For any $t, r \in I$, by Comparative Richness, we can find $x_t, x_r \in X$ such that $x_t \approx^f A_t$, $x_r \approx^f A_r$, and $\Omega = \{\{x_t, x_r\}\} \cup \{A_k\}_{k \in I}$ is an MIC. Clearly, $A_t \bowtie_\Omega \{x_t\}$ and $A_r \bowtie_\Omega \{x_r\}$. If $A_t \trianglerighteq^f A_r$, then $x_t \in f(\{x_t, x_r\})$, which implies $\{x_t\} \trianglerighteq_\Omega \{x_r\}$. Thus, $A_t \trianglerighteq_\Omega A_r$. If $A_t \triangleright^f A_r$, then $x_r \notin f(\{x_t, x_r\})$, which implies $\{x_t\} \triangleright_\Omega \{x_r\}$. Thus, $A_t \triangleright_\Omega A_r$. Since $\trianglerighteq_\Omega$ and $\trianglerighteq^f$ agree on $\{A_t, A_r\}$, so do $\trianglerighteq_\Pi$ and $\trianglerighteq^f$. We are done.

Finally, we show statement (3). Consider two menus A and B with $B \subseteq A$. We first show that $A \trianglerighteq^f B$. WLOG, assume $A = B \cup \{z\}$ for some $z \in X \backslash B$. Suppose to the contrary that $B \triangleright^f A$. By Comparative Richness, we can find $x \in X$ such that $x \perp A$ and $x \approx^f A$ (i.e., $A \bowtie^f \{x\}$). Since $B \triangleright^f A$, we have $B \triangleright^f \{x\}$, and thus $x \notin f(B \cup \{x\})$. However, $x \in f(A \cup \{x\})$ contradicts $x \perp z$.

Next, assume $A \bowtie^f B$, and we show $f(B) \subseteq f(A)$. WLOG, assume $A = B \cup \{z\}$ for some $z \in X \backslash B$. Suppose to the contrary that there is $x \in f(B) \backslash f(A)$. By

Comparative Richness, there is $y \in X$ such that $y \perp A$ and $\{y\} \bowtie^f A \bowtie^f B$. By statement (2), we have $y \in f(A \cup \{y\}) \cap f(B \cup \{y\})$. By statement (1) of Theorem 1, $x \in f(B \cup \{y\}) \backslash f(A \cup \{y\})$. Since $y \perp z$, we have $f(B \cup \{y\}) \subseteq f(A \cup \{y\})$, which contradicts $x \in f(B \cup \{y\}) \backslash f(A \cup \{y\})$. □

Proof of Proposition 3. Consider the $\trianglerighteq$-canonical extension (Y, g) of (X, f). Let $\Phi : Y \backslash X \to \mathcal{M}(X)$ be the bijection defined in Definition 4. We first argue that for all $x \in Y$ and $y \in Y \backslash X$, we have $x \perp y$. WLOG, assume $x \neq y$. By the construction of g, we have for all $A \in \mathcal{M}(Y)$, either $g(A \cup \{y\}) = g(A)$, or $g(A \cup \{y\}) = g(A) \cup \{y\}$, or $g(A \cup \{y\}) = \{y\}$. Thus, if $x \in A$ and $x \in g(A \cup \{y\})$, then $x \in g(A) \subseteq g(A \cup \{y\})$. Inversely, consider $A \in \mathcal{M}(Y)$ such that $x \notin A$, $y \in A$, and $y \in g(A \cup \{x\})$. By $y \in g(A \cup \{x\})$, we have $\Phi(y) \trianglerighteq (A \cup \{x\}) \cap X$ (if $(A \cup \{x\}) \cap X \neq \emptyset$) and for all $z \in (A \cup \{x\}) \backslash X$, $\Phi(y) \trianglerighteq \Phi(z)$. By monotonicity of $\trianglerighteq$, we have $\Phi(y) \trianglerighteq A \cap X$ (if $A \cap X \neq \emptyset$), and thus $y \in g(A)$. What remains to be shown is that $g(A) \subseteq g(A \cup \{x\})$. Note that for all $z \in g(A) \backslash X$, we have $\Phi(z) \bowtie \Phi(y)$, and thus $z \in g(A \cup \{x\})$. If $g(A) \cap X = \emptyset$, then we are done. If $g(A) \cap X \neq \emptyset$, then $A \cap X \bowtie \Phi(y) \trianglerighteq (A \cup \{x\}) \cap X$. By monotonicity of $\trianglerighteq$, we have $A \cap X \bowtie \Phi(y) \bowtie (A \cup \{x\}) \cap X$, and thus $f(A \cap X) \subseteq f((A \cup \{x\}) \cap X)$. Since $g(A) \cap X = f(A \cap X)$ and $g(A \cup \{x\}) \cap X = f((A \cup \{x\}) \cap X)$, we have $g(A) \subseteq g(A \cup \{x\})$. The claim that $x \perp y$ is shown.

To see that Comparative Richness holds for g, consider $A, B \in \mathcal{M}(Y)$ with $B \subseteq A$. If there is $x \in g(B) \backslash X$, then $x \approx^g B$ and $x \perp A$. If $g(B) \subseteq X$, then let $x \in Y \backslash X$ satisfy $\Phi(x) = B \cap X$, and we have $g(B \cup \{x\}) = g(B) \cup \{x\}$. Thus, $x \approx^g B$ and $x \perp A$. The Comparative Richness of g is shown. That $\trianglerighteq^g$ coincides with $\trianglerighteq$ on $\mathcal{M}(X)$ is evident by the construction of the canonical extension. □

Proof of Theorem 3. Clearly, statement (4) implies (1), (2), and (3).

(5) ⇒ (4): We argue that (5) indicates that $x \in f(A)$ if and only if for all $y \in A$, $\{x\} \trianglerighteq^f \{y\}$—i.e., the preference $\succsim$ that agrees with $\trianglerighteq^f$ restricted on singleton menus rationalizes f. To see this, assume first that $x \in f(A)$. By (5), we have for all $y \in A$, $\{x\} \bowtie^f A \trianglerighteq^f \{y\}$ (by monotonicity of $\trianglerighteq^f$). Inversely, if for all $y \in A$, $\{x\} \trianglerighteq^f \{y\}$, then pick some $y' \in f(A)$, and we have $\{x\} \trianglerighteq^f \{y'\} \bowtie^f A$. By monotonicity of $\trianglerighteq^f$, $\{x\} \bowtie^f A$ and thus $x \in f(A)$.

(1), (2), or (3) ⇒ (5): In the remaining proof, we fix a menu A and an alternative $x \in f(A)$. We also fix some $z \in X$ with $z \perp A$ and $A \approx^f \{z\}$. The selection of z ensures $z \bowtie^f A$. First, assume statement (1). Since $x \in f(A)$, we have $x \in f(A \cup \{z\})$. Since $z \in f(A \cup \{z\})$, by Binary Sen's α, we have $f(\{x, z\}) =$

$\{x, z\}$. Thus, $\{x\} \bowtie^f \{z\} \bowtie^f A$. Second, assume statement (2). Consider $w \in X$ such that $w \perp A \cup \{z\}$ and $\{x\} \bowtie^f \{w\}$. Since $f(\{x, w\}) = \{x, w\}$ and $x \in f(A \cup \{z, w\})$, by Binary Sen's β, we have $w \in f(A \cup \{z, w\})$. Thus, $\{x\} \bowtie^f \{w\} \bowtie^f \{z\} \bowtie^f A$. Finally, assume statement (3). For all $y \in A$, $z \in f(\{y, z\})$. Thus, $z \in \mathcal{W}(A \cup \{z\}) \neq \emptyset$, and we have $f(A \cup \{z\}) = \mathcal{W}(A \cup \{z\})$. Since $x \in f(A \cup \{z\})$, $x \in \mathcal{W}(A \cup \{z\})$, which further implies $x \in f(\{x, z\})$. It follows that $\{x\} \bowtie^f \{z\} \bowtie^f A$. □

Proof of Theorem 4. It remains to show that statement (2) implies (3). Reducibility implies that we can remove unchosen alternatives sequentially such that for every menu A, $f(f(A)) = f(A)$. To see $A \bowtie^f f(A)$, consider $z \in X$ such that $\{z\} \bowtie^f A$ and $z \perp A$. Since $f(A \cup \{z\}) = f(A) \cup \{z\}$, we have $f(f(A) \cup \{z\}) = f(A) \cup \{z\}$. Therefore, $f(A) \bowtie^f \{z\} \bowtie^f A$. □

Proof of Theorem 5. It remains to show that statement (1) implies (3). Let (X, f) satisfy Comparative Richness and WrARP. Define preference $\succsim$ over X such that for all $x, y \in X$, $x \succsim y$ if and only if $\{x\} \trianglerighteq^f \{y\}$.

We show that if $x \in f(A)$ and $A \rhd^f \{x\}$, then for all $y \in X$ with $\{y\} \bowtie^f A$, $y \in J^+_{\succsim}(x)$. Suppose to the contrary that we can find $x \in f(A)$ and $y, z \in X$ such that $\{y\} \bowtie^f A \rhd^f \{z\} \rhd^f \{x\}$. Comparative Richness ensures that we can assume $y \perp z$ and $\{y, z\} \perp A$. It follows that $x \notin f(\{x, z\})$, $x \in f(A \cup \{z\})$, and $z \notin f(A \cup \{z\})$, which contradicts WrARP.

Next, we construct γ. For any menu A, if $x \in A$ satisfies $J^+_{\succsim}(x) = \emptyset$, let $\gamma(x, A) = x$. For $x \in A$ with $J^+_{\succsim}(x) \neq \emptyset$, (i) if $A \bowtie^f \{x\}$, then let $\gamma(x, A) = x$; (ii) if $A \bowtie^f \{y\}$ for some $y \in J^+_{\succsim}(x)$ and $x \notin f(A)$, then let $\gamma(x, A) = x$; (iii) otherwise, let $\gamma(x, A) = y$ for some $y \in J^+_{\succsim}(x)$.

To see that $(\succsim, \gamma)$ satisfies condition (2) in Definition 5, note that if $x \in A \subseteq B$ and $\gamma(x, A) = y \in J^+_{\succsim}(x)$, then we have either (i) $A \rhd^f \{y\}$ or (ii) $A \bowtie^f \{y\}$ and $x \in f(A)$. Thus, either $B \rhd^f \{y\}$, which directly implies $\gamma(x, B) \in J^+_{\succsim}(x)$, or $B \bowtie^f A \bowtie^f \{y\}$. By monotonicity of $\trianglerighteq^f$, the latter case implies $x \in f(B)$, and thus $\gamma(x, B) \in J^+_{\succsim}(x)$. To see that $(\succsim, \gamma)$ satisfies condition (3) in Definition 5, assume to the contrary that there are menus A and B and $x, y \in A \cap B$ with $x \sim y$, $\gamma(x, A) = x$, $\gamma(y, B) = y$, $\gamma(x, B) \in J^+_{\succsim}(x)$, and $\gamma(y, A) \in J^+_{\succsim}(y)$. By the construction of γ, we have $A \bowtie^f B \bowtie^f \{z\}$ for some $z \in J^+_{\succsim}(x)$, $f(A) \cap \{x, y\} = \{y\}$, and $f(B) \cap \{x, y\} = \{x\}$. This contradicts WrARP.

Finally, note that our construction ensures that for every menu A and every $x \in A$, $A \trianglerighteq^f \{\gamma(x, A)\}$, and $x \in f(A)$ if and only if $A \bowtie^f \{\gamma(x, A)\}$. Thus,

$(\succsim, \gamma)$ MO-rationalizes f. $\square$

Proof of Theorem 6. To see the necessity, suppose that f is GA-rationalized by $(\Gamma, \tau, \succsim)$. For a given menu A, consider $x \in \mathcal{U}_{\succsim}(A)$. We argue that x satisfies the statement of WARP(LA)*. To see this, consider menu B such that $f(B) \cap A \neq \emptyset$. If $x \notin \Gamma(B)$, then $\Gamma(B \backslash \{x\}) = \Gamma(B)$, and we have $f(B) = f(B \backslash \{x\})$. Thus, $f(B) \neq f(B \backslash \{x\})$ implies $x \in \Gamma(B)$. By $f(B) \cap A \neq \emptyset$, there exists $y \in A$ such that $y \succsim \tau(\Gamma(B))$. Thus, we have $x \succsim \tau(\Gamma(B))$ and $x \in f(B)$.

For the sufficiency part, define xPy if there exists menu A such that $x, y \in A$, $x \in f(A)$, $y \notin f(A)$, and $f(A) \neq f(A \backslash \{y\})$. Following the proof of Lemma 1 in MNO, we can show that P is acyclic. By the Szpilrajn extension theorem, there is a preference $\succsim$ over X such that xPy implies $x \succ y$. For each menu A, define $\Gamma(A) = f(A) \cup \{x \in A : \forall y \in f(A), y \succ x\}$. To see that Γ is an attention filter, note that for every menu A and every $z \in A \backslash \Gamma(A)$, we have $z \notin f(A)$ and $z \succsim x$ for some $x \in f(A)$. Therefore, we should have $f(A \backslash \{z\}) = f(A)$ since otherwise, we have for all $x \in f(A)$, xPz, which is a contradiction. The fact that $z \succsim x$ for some $x \in f(A)$ and $f(A) = f(A \backslash \{z\})$ implies $\Gamma(A) = \Gamma(A \backslash \{z\})$.

Finally, we construct τ. For every menu B such that $B = \Gamma(A)$ for some menu A, let $\tau(B)$ be an arbitrary element of $\{x \in f(A) : \forall y \in f(A), y \succsim x\}$. For any other menu B, define $\tau(B)$ arbitrarily. Note that τ is well defined, since for any two menus A and A' with $\Gamma(A) = \Gamma(A') = B$, we have $f(A) = f(B) = f(A')$. Clearly, f is GA-rationalized by $(\succsim, \Gamma)$. $\square$

Proof of Theorem 7. Since OA-rationalizability implies GA-rationalizability, it remains to show that GA-rationalizability implies OA-rationalizability. Consider a tuple $(\succsim, \Gamma, \tau)$ and assume that it GA-rationalizes f. By the proof of Theorem 6, assume that for every menu A, $\Gamma(A) = f(A) \cup \{y \in A : \forall x \in f(A), x \succ y\}$. First, we show that $\Gamma(A) \bowtie^f A$. To see this, consider $z \in X$ such that $z \perp A$ and $\{z\} \bowtie^f A$. We have $f(A \cup \{z\}) = f(A) \cup \{z\}$, and thus $\Gamma(A \cup \{z\}) \subseteq \Gamma(A) \cup \{z\}$. It follows that $\Gamma(A \cup \{z\}) \cap A \subseteq \Gamma(A)$. Since $f(\Gamma(A \cup \{z\})) = f(A \cup \{z\})$, we have $(\Gamma(A \cup \{z\}) \cap A) \bowtie^f \{z\} \bowtie^f A$. By monotonicity of $\trianglerighteq^f$, we have $\Gamma(A) \bowtie^f A$.

We proceed to show that Γ is an ordered attention filter, and we are done. For two distinct menus C and D, define $C \rhd^* D$ if one of the following occurs:

(i) $C = \Gamma(C)$ and $D \neq \Gamma(D)$; (ii) $C = \Gamma(C)$, $D = \Gamma(D)$, and $C \rhd^f D$;

(iii) $C = \Gamma(C)$, $D = \Gamma(D)$, $C \bowtie^f D$, and $f(D) \subsetneq f(C)$;

(iv) $C = \Gamma(C)$, $D = \Gamma(D)$, $C \bowtie^f D$, $f(C) = f(D)$, and $D \subsetneq C$.

For any menu A, we show that $B \in \mathcal{M}(A)$ and $B \neq \Gamma(A)$ implies $\Gamma(A) \rhd^* B$. To see this, note that since $\Gamma(\Gamma(A)) = \Gamma(A) \bowtie^f A \trianglerighteq^f B$, we only need to consider the case in which $\Gamma(B) = B \bowtie^f \Gamma(A)$. The monotonicity of $\trianglerighteq^f$ further implies $f(B) \subseteq f(A) = f(\Gamma(A))$. Thus, we can further assume $f(B) = f(A)$. We have

$$\begin{aligned} B = \Gamma(B) &= f(B) \cup \{x \in B : \forall y \in f(B), y \succ x\} \\ &= f(A) \cup \{x \in B : \forall y \in f(A), y \succ x\} \\ &\subseteq f(A) \cup \{x \in A : \forall y \in f(A), y \succ x\} = \Gamma(A) \end{aligned}$$

Since $B \neq \Gamma(A)$ and $B \subseteq \Gamma(A)$, by (iv), we have $\Gamma(A) \rhd^* B$. Our construction ensures that the binary relation $\rhd^*$ is acyclic. Thus, there is a linear order $\trianglerighteq$ over menus that extends $\rhd^*$, with which we have $\Gamma_{\trianglerighteq} = \Gamma$. □

Proof of Theorem 8. (3) ⇒ (2): For simplicity, let $S = T + 1$. We construct a preference $\succsim^S$ over X^S as follows. For any two vectors of alternatives $(x_k)_{k=1}^S$ and $(y_k)_{k=1}^S$, let $A' = \cup_{k=1}^S \{x_k\}$ and $B' = \cup_{k=1}^S \{y_k\}$. Define any $\succsim^S$ such that (i) when $x_1 \in f(A')$ and $y_1 \notin f(B')$, we have $(x_k)_{k=1}^S \succ^S (y_k)_{k=1}^S$, and (ii) when $x_1 \in f(A')$ and $y_1 \in f(B')$, we have $(x_k)_{k=1}^S \succsim^S (y_k)_{k=1}^S$ if and only if $A' \trianglerighteq^f B'$. We argue that any $\succsim^S$ that satisfies (i) and (ii) S-rationalizes f. To see this, consider an arbitrary menu A. On the one hand, if x is chosen in A, then by statement (3), we can find some submenu B of A such that x is chosen in B and $A \bowtie^f B$ with $B = \{x, x_1, ..., x_k\}$ for some $k \leq T$. Our construction of $\succsim^S$ and the monotonicity of $\trianglerighteq^f$ ensures that the vector $(x, x_1, ..., x_k, x, ..., x) \in X^S$ maximizes $\succsim^S$ in A^S. On the other hand, if some vector $(x, x_1, ..., x_T)$ maximizes $\succsim^S$ in A^S, then the construction of $\succsim^S$ implies $x \in f((\cup_{k=1}^T \{x_k\}) \cup \{x\})$ and $(\cup_{k=1}^T \{x_k\}) \cup \{x\} \bowtie^f A$. By the monotonicity of $\trianglerighteq^f$, we have $x \in f(A)$.

(2) ⇒ (1): Assume that f is S-rationalized by $\succsim^S$ over X^S, and consider an arbitrary $x \in f(A)$. There is a vector of alternatives $(x_k)_{k=1}^S \in \mathcal{U}_{\succsim^S}(A^S)$ such that $x_1 = x$. Thus, for any $y \neq x$, $x \notin f(A \backslash \{y\})$ implies $y \in C = \cup_{k=2}^S \{x_k\}$. For any menu B with $C \subseteq B \subseteq A$, $(x_k)_{k=1}^S \in \mathcal{U}_{\succsim^S}(A^S)$ implies $x \in f(B)$. Since $|C| \leq T$, T-Weak Sen's α holds.

(1) ⇒ (3): Consider an arbitrary $x \in f(A)$. Pick $z \in X$ with $z \perp A$ and $\{z\} \bowtie^f A$. If $z = x$, then we are done. Consider the case in which $z \neq x$. Note that $x, z \in f(A \cup \{z\})$. Define $C = \{y \neq x : x \notin f((A \cup \{z\}) \backslash \{y\})\}$. Since $x \perp z$, $z \notin C$. By statement (1), we have $|C| \leq T$ and $x \in f(C \cup \{x\} \cup \{z\})$. Since $z \perp C \cup \{x\}$, $C \cup \{x\} \bowtie^f \{z\}$. Therefore, $C \cup \{x\} \bowtie^f A$ and $x \in f(C \cup \{x\})$. □

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Online Appendix (for online publication only)

In this note, we list a series of examples and results that we omitted in the main text.

Examples related to MP-rationalizability

Our first example shows that MP-rationalizability and Comparative Richness together cannot imply rationalizability.

Example 1. Let $X = \{x_1, x_2, x_3, y_1, y_2, y_3, z\}$. Consider a linear order $\trianglerighteq$ over $\mathcal{M}(X)$ that satisfies $\{x_1, x_2, x_3, z\} \vartriangleright \{x_1, x_2, x_3\} \vartriangleright \{z\} \vartriangleright \{x_1, y_1\} \vartriangleright \{x_1\} \vartriangleright \{y_1\} \vartriangleright \{x_2, y_2\} \vartriangleright \{x_2\} \vartriangleright \{y_2\} \vartriangleright \{x_3, y_3\} \vartriangleright \{x_3\} \vartriangleright \{y_3\} \vartriangleright A$ for every other menu A. Let f be MP-rationalized by $\trianglerighteq$. Clearly, f is not rationalizable, since $f(\{x_1, x_2, x_3\}) = \{x_1, x_2, x_3\}$ while $f(\{x_1, x_2\}) = \{x_1\}$. Note that f satisfies Comparative Richness with $\{\{x_1, x_2, x_3\}, \{y_1\}, \{y_2\}, \{y_3\}, \{z\}\}$ being an MIC and the shadow preference $\trianglerighteq^f$ satisfying $\{x_1, x_2, x_3\} \bowtie^f \{z\} \vartriangleright^f \{x_1\} \bowtie^f \{x_1, x_2\} \bowtie^f \{x_1, x_3\} \bowtie^f \{y_1\} \vartriangleright^f \{x_2\} \bowtie^f \{x_2, x_3\} \bowtie^f \{y_2\} \vartriangleright^f \{x_3\} \bowtie^f \{y_3\}$. □

The next example demonstrates that without Comparative Richness, Aizerman's axiom alone cannot imply MP-rationalizability.

Example 2. Consider $X = \{x, y, z\}$ and a choice correspondence f such that

$$f(\{x, y\}) = \{x\}, \; f(\{y, z\}) = \{y\}, \; f(\{x, z\}) = \{z\}, \; f(\{x, y, z\}) = \{x, y, z\}.$$

This choice correspondence clearly satisfies Aizerman's axiom. However, it fails to be MP-rationalizable. To see this, suppose to the contrary that there exists an implied menu preference $\trianglerighteq$ MP-rationalizing f. The choices made in binary menus imply that $\{x\} \vartriangleright \{y\}$, $\{y\} \vartriangleright \{z\}$, and $\{z\} \vartriangleright \{x\}$, which is a contradiction. □

Examples related to MO-rationalizability

In our next example, we present a choice correspondence that is MO-rationalizable and satisfies Comparative Richness but fails to be rationalizable.

Example 3. Let $X = \{x, y, z, w\}$ and consider a monotone overevaluation system $(\succsim, \gamma)$ over X such that $w \succ x \sim y \sim z$, and for every menu A, (1) if $z \in A$, then $\gamma(z, A) = z$; (2) if $w \in A$, then $\gamma(w, A) = w$; (3) if $A \cap \{x, y\} = \{x\}$, then $\gamma(x, A) = x$; (4) if $A \cap \{x, y\} = \{y\}$, then $\gamma(y, A) = y$; and (5) if $A \cap \{x,y\} = \{x,y\}$, then $\gamma(x,A) = \gamma(y,A) = w$. Let f be the choice correspondence that is MO-rationalized by $(\succsim, \gamma)$. It can easily be shown that f satisfies Comparative Richness with $\{\{x,y\},\{z\},\{w\}\}$ being an MIC and $\{x,y\} \bowtie^f \{w\} \rhd^f \{z\} \bowtie^f \{x\} \bowtie^f \{y\}$. Clearly, f is not rationalizable since $f(\{x,w\}) = \{w\}$ and $f(\{x,y,w\}) = \{x,y,w\}$. □

The next example demonstrates that without Comparative Richness, WrARP alone is not sufficient for MO-rationalizability.

Example 4. Let $X = \{x,y,z,w,r\}$ and consider a choice correspondence f with

$$f(\{x,y\}) = \{x\}, f(\{x,y,z,w\}) = \{x,z,w\},$$

and for every other menu A, $f(A) = A$. This choice correspondence satisfies WrARP, since only x, z, and w are revealed to be strictly preferred to y but y is not revealed to be strictly preferred to any alternative. However, f cannot be MO-rationalizable. To see this, suppose to the contrary that there exists a monotone overevaluation system $(\succsim, \gamma)$ that MO-rationalizes f. Since $y \notin f(\{x,y\})$ but $y \in f(\{x,y,z\})$, we can infer that $\gamma(y,\{x,y\}) = y$ and $\gamma(y,\{x,y,z\}) \in J^+_{\succsim}(y)$. By a similar argument, we have $\gamma(y,\{x,y,z,w\}) = y$ and $\gamma(y,\{x,y,z,w,r\}) \in J^+_{\succsim}(y)$. However, $\gamma(y,\{x,y,z\}) \in J^+_{\succsim}(y)$ and $\gamma(y,\{x,y,z,w\}) = y$ together violate condition (2) of Definition 5. □

A crucial observation from Example 4 is that for any two alternatives x and y, WrARP alone allows for multiple rounds of violations of the WARP between x and y when the choice menu enlarges. Specifically, consider four menus $A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4$ with $x, y \in A_1$. WrARP alone does not rule out the possibility that x is chosen in all four menus and y is only chosen in A_2 and A_4. In this case, the WARP is violated between x and y for the menu pair A_1 and A_2 and the menu pair A_3 and A_4. However, together with Comparative Richness, WrARP does not allow for such a choice pattern.

Examples related to limited attention

Next, we present an example that demonstrates that without Comparative Richness, GA-rationalizability is insufficient for OA-rationalizability.

Example 5. Let $X = \{x,y,z,w\}$ and consider a choice correspondence f with

$$f(\{x,y,z,w\}) = \{x,y,z\}, f(\{x,y,z\}) = \{x\},$$

$$f(\{x,y,w\}) = \{x,w\}, f(\{y,z,w\}) = \{y,w\}, f(\{x,z,w\}) = \{z,w\},$$

and for every other menu A, $f(A) = A$. First, we show that f is not OA-rationalizable. Suppose to the contrary that f is OA-rationalized by $(\succsim, \Gamma_{\trianglerighteq}, \tau)$. Since $f(\{x,y,z,w\}) = \{x,y,z\} \neq f(\{x,y,z\})$, we have $x \succ w$, $y \succ w$, and $z \succ w$. It follows that $f(\{x,y,w\}) = \{x,w\}$ implies $\Gamma_{\trianglerighteq}(\{x,y,w\}) = \{x,w\}$, $f(\{y,z,w\}) = \{y,w\}$ implies $\Gamma_{\trianglerighteq}(\{y,z,w\}) = \{y,w\}$, and $f(\{x,z,w\}) = \{z,w\}$ implies $\Gamma_{\trianglerighteq}(\{x,z,w\}) = \{z,w\}$. However, those attention sets imply $\{x,w\} \triangleright \{y,w\} \triangleright \{z,w\} \triangleright \{x,w\}$, which is a contradiction.

Next, we show that (X, f) is GA-rationalizable. Let $\succsim$ be such that $x \succ y \succ z \succ w$. Define the attention filter Γ such that $\Gamma(\{x,y,w\}) = \{x,w\}$, $\Gamma(\{y,z,w\}) = \{y,w\}$, $\Gamma(\{x,z,w\}) = \{z,w\}$, and for every other menu A, $\Gamma(A) = A$. Define the threshold function τ such that $\tau(\{x,y,z,w\}) = z$, $\tau(\{x,y,z\}) = x$, and for every other menu A, $\tau(A)$ is the alternative in A that ranks the lowest according to $\succsim$. We can verify that the tuple $(\succsim, \Gamma, \tau)$ GA-rationalizes f. □

Examples and results related to T-rationalizability

The next example demonstrates that without Comparative Richness, Weak Sen's α alone does not imply 2-rationalizability.

Example 6. Let $X = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$. For each $k \in \{1,2,3,4\}$, let $A_k = \{x_k, y_k\}$, and let $A_5 = A_1$ and $A_6 = A_2$ (alternatives x_5, y_5, x_6, and y_6 are defined accordingly). Consider a choice correspondence f that is defined as follows. For each $k \in \{1,2,3,4\}$ and menu A with $A \subseteq A_k \cup A_{k+1}$, (1) if $A_k \subseteq A$, then $f(A) = \{x_k\}$; (2) if $A_k \nsubseteq A$ and $A_{k+1} \subseteq A$, then $f(A) = \{x_{k+1}\}$; (3) if $A_k \nsubseteq A$, $A_{k+1} \nsubseteq A$, and $A \cap \{x_k, x_{k+1}\} \neq \emptyset$, then $f(A) = A \cap \{x_k, x_{k+1}\}$; and (4) if $A_k \nsubseteq A$, $A_{k+1} \nsubseteq A$, and $A \cap \{x_k, x_{k+1}\} = \emptyset$, then $f(A) = A \cap \{y_k, y_{k+1}\}$. For every other menu B, $f(B) = \bigcup_{k \in I_B}(B \cap A_k)$, where $I_B = \{k \in \{1,2,3,4\} : B \cap A_{k+2} \neq \emptyset\}$.

For the choice correspondence f, consider a menu A and two distinct alternatives $x, y \in A$ such that $x \in f(A)$ and $x \notin f(A \backslash \{y\})$. There are two possible cases: Either (1) $x \in A_k$ and $y \in A_{k+2}$ for some $k \in \{1,2,3,4\}$, or (2) $A = A_k \cup A_{k+1}$ for some $k \in \{1,2,3,4\}$ with $x = x_k$ and $y = y_k$. In both cases, for all menu B contained in A with $x, y \in B$, we have $x \in f(B)$. Thus, f satisfies Weak Sen's α. However, we can easily show that if there is a preference $\succsim^2$ over X^2 that 2-rationalizes f, then we have $(x_1, y_1) \succ^2 (x_2, y_2) \succ^2 (x_3, y_3) \succ^2 (x_4, y_4) \succ^2 (x_1, y_1)$, which is a contradiction. □

Now, we formally define T-Weak Sen's β and demonstrate that even with Comparative Richness, T-Weak Sen's α is not equivalent to T-Weak Sen's β.

Axiom (T-Weak Sen's β)**.** *For all $A, B, C \in \mathcal{M}(X)$ and $x \in X$, with $x \notin C$, $C \subseteq B \subseteq A$, and $|C| = T$, if $x \in f(B) \cap f(A)$ and for all $y \in C$, $x \in f(A \backslash \{y\}) \not\subseteq f(A)$, then $f(B) \subseteq f(A)$.*

Example 7. Let $X = \{x_k\}_{k=1}^{\infty}$. Let $\succsim$ and $\succsim'$ be two linear orders on X such that $x_k \succsim x_l$ if and only if $x_l \succsim' x_k$ if and only if $k \geq l$. Consider a choice correspondence f such that $f(A) = \mathcal{U}_{\succsim}(A)$ when $|A|$ is odd, and $f(A) = \mathcal{U}_{\succsim'}(A)$ when $|A|$ is even. Consider a quasi-shadow preference of f, denoted $\trianglerighteq$, such that $A \trianglerighteq B$ if and only if $|A| \geq |B|$. Let $(\bar{X}, \bar{f})$ be the $\trianglerighteq$-canonical extension of (X, f). Clearly, for all distinct $x, y \in \bar{X}$, if $\{x, y\} \not\subseteq X$, then $x \perp y$. We show that $(\bar{X}, \bar{f})$ satisfies 1-Weak Sen's β, which implies that it satisfies T-Weak Sen's β for all $T \in \mathbb{N}_+$. Pick menus A, B and distinct x, y with $x, y \in B \subseteq A$. Suppose $x \in \bar{f}(A) \cap \bar{f}(B)$ and $x \in \bar{f}(A \backslash \{y\}) \not\subseteq \bar{f}(A)$. This can only happen if $x \not\perp y$, which implies $x, y \in X$. Furthermore, by the definition of f, $x \in \bar{f}(A) \cap \bar{f}(A \backslash \{y\})$ implies $A \cap X = \{x, y\}$, and thus for all $z \in A \backslash X$, $\{x\} \trianglerighteq^{\bar{f}} \{z\}$. Since the quasi-shadow preference satisfies $\{x, y\} \triangleright \{x\}$, we have for all $z \in A \backslash X$, $\{x, y\} \triangleright^{\bar{f}} \{z\}$. Thus, $\bar{f}(B) = \bar{f}(A) = \bar{f}(\{x, y\}) = \{x\}$. It follows that $\bar{f}$ satisfies 1-Weak Sen's β and thus T-Weak Sen's β for all $T \geq 2$. However, for any $T \geq 2$, f is not T-rationalizable, since the quasi-shadow preference $\trianglerighteq$ (i.e., $\trianglerighteq^{\bar{f}}$) cannot satisfy statement (3) in Theorem 8. $\square$

Next, we present the result omitted in the "Choice with bounded disappointment aversion" example. Let X be the set of simple lotteries over $\mathbb{R}$—i.e., each lottery $x \in X$ takes the form of $x = (x_1, p_1; ...; x_n, p_n)$ with the interpretation that x yields monetary payoff x_k with probability p_k for each $k \in \{1, ..., n\}$. Consider the utility function $u : X^2 \to \mathbb{R}$ such that for any $x, y \in X$ with $x = (x_1, p_1; ...; x_n, p_n)$ and $y = (y_1, q_1; ...; y_m, q_m)$, we have

$$u(x, y) = \sum_{k=1}^{n} p_k v(x_k) - \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j \lambda \min\{\max\{v(y_j) - v(x_i), 0\}, L\},$$

where $L > 0$, $\lambda \in (0, 1)$, and $v : \mathbb{R} \to \mathbb{R}$ is strictly increasing and satisfies $v(\mathbb{R}) = \mathbb{R}$.

Let f be a choice correspondence that is 2-rationalized by u defined above. The following proposition shows that f accommodates the phenomenon of preference reversal—i.e., for two lotteries the DM attaches the same certainty equivalent to, she strictly prefers the low-risk lottery to the high-risk one.

Proposition. *Let f be a choice correspondence that is 2-rationalized by u defined above. Consider three lotteries $x = (a, p; 0, 1 - p)$, $y = (b, q; 0, 1 - q)$, and $z = (c, 1)$ such that $a > b > c > 0$, $q > p$, and $L \geq v(a) - v(0)$. If $f(\{x, z\}) = \{x, z\}$ and $f(\{y, z\}) = \{y, z\}$, then $f(\{x, y\}) = \{y\}$.*

Proof. First note that $u(z, z) > u(z, x)$ and $u(z, z) > u(z, y)$. Thus, the DM evaluates z according to $u(z, z)$ in both $\{x, z\}$ and $\{y, z\}$. By $f(\{x, z\}) = \{x, z\}$, we have $pv(a) + (1 - p)v(0) > v(c)$, since otherwise $u(x, x) - u(z, z) < 0$ and $u(x, z) - u(z, z) < 0$. Thus $u(x, z) - u(x, x) = \lambda(1-p)[pv(a) + (1-p)v(0) - v(c)] > 0$, and $f(\{x, z\}) = \{x, z\}$ implies $u(x, z) = u(z, z)$. By a similar argument, we have $qv(b) + (1 - q)v(0) > v(c)$ and $u(y, z) = u(z, z)$. The condition $u(x, z) = u(y, z) = u(z, z)$ implies $\Delta := pv(a) + (1 - p)v(0) - qv(b) - (1 - q)v(0) = \lambda(q - p)[v(c) - v(0)] > 0$. Next, observe that $u(y, y) - u(y, x) = \lambda(1 - q)\Delta + \lambda pq[v(a) - v(b)] > 0$ and $u(x, y) - u(x, x) = \lambda(1 - p)\Delta > 0$. Thus, the DM's choice in $\{x, y\}$ is determined by the sign of $u(x, y) - u(y, y)$. Since $u(x, y) - u(y, y) = \lambda(q - p)[v(c) - (1 - q)v(0) - qv(b)] < 0$, $f(\{x, y\}) = \{y\}$. $\square$